Aproximação de campo-médio para otimização de tráfego

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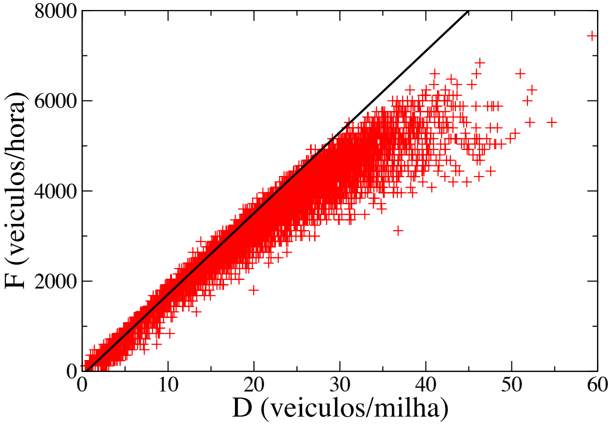
Sumário

Motivação

"The volume of vehicular traffic in the past several years has rapidly outstripped the capacities of the nation's highways. It has become increasingly necessary to understand the dynamics of traffic flow and obtain a mathematical description of the process."

Greenberg 1959

Diagrama fundamental do tráfego



Sumário

Modelo finito (Alimohammadi *et al* 2001)

$$\bullet \rightarrow \text{ocupado};\ \circ \rightarrow \text{vazio}$$

$$E_n = P(\overbrace{\circ \ldots \circ}^{n})$$

Método de intervalos vazios

$$\begin{aligned}
\circ\bullet \quad &\rightarrow (\bullet\bullet,\ \bullet\circ, \circ\circ) \\
\bullet\circ \quad &\rightarrow (\bullet\bullet,\ \circ\bullet, \circ\circ) \\
\bullet\bullet \quad &\rightarrow (\bullet\circ,\ \circ\circ, \circ\bullet) \\
\circ\circ \quad &\rightarrow (\bullet\circ,\ \bullet\bullet, \circ\bullet)
\end{aligned}$$

Equação mestra

$$
\begin{aligned}
\frac{dE_n}{dt} = \; & D_e P(\circ \overbrace{\bullet \circ \ldots \circ}^{n}) + D_d P(\overbrace{\circ \ldots \circ \bullet}^{n} \circ) + \\
& - D_d P(\bullet \overbrace{\circ \circ \ldots \circ}^{n}) - D_e P(\overbrace{\circ \ldots \circ \circ}^{n} \bullet) + \\
& - R_d P(\bullet \overbrace{\circ \circ \ldots \circ}^{n}) - R_e P(\overbrace{\circ \ldots \circ \circ}^{n} \bullet) + \\
& + C_e P(\bullet \overbrace{\bullet \circ \ldots \circ}^{n}) + C_d P(\overbrace{\circ \ldots \circ \bullet}^{n} \bullet).
\end{aligned}
$$

$$\frac{dE_n}{dt} = E_{n-1} - 2E_n + E_{n+1} - \gamma(E_n - E_{n+1})$$

onde $\gamma = {}^{R}/_{D}$

Condição de fronteira

$$E_0(t) = 1,\ E_{L+1} = 0$$

Solução estacionária

$$E_n^* = \frac{1}{1-(1+\gamma)^{-L-1}}(1+\gamma)^{-n} + \frac{-(1+\gamma)^{-L-1}}{1-(1+\gamma)^{-L-1}}.$$

$$\lim_{L\to\infty} \langle n_i \rangle (\gamma) \simeq \frac{\gamma}{2}, \text{ e}$$

$$\lim_{L\to\infty} \langle n_i n_{i+1} \rangle (\gamma) \simeq \frac{\gamma^2}{2}.$$

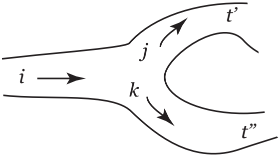
Sumário

Otimização local de fluxos

Equilíbrio de tráfego

$$t_i = \bar{t}_i + a j_i + (b j_i^2 + c j_i^3 + \ldots)\Theta(j_i - j_c)$$

$$C(\mathbf{j}) = \sum_{i=1}^{n} \left(a_i j_i + b_i j_i^2\right)$$



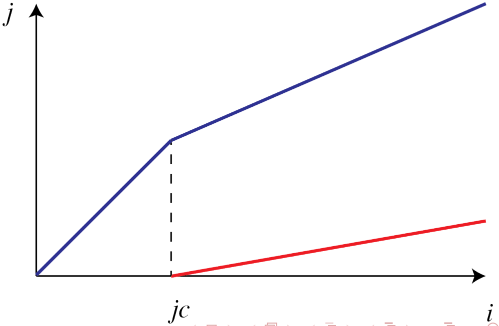
Otimização local de tráfego.

$$C(\mathbf{j}) = \sum_{i=1}^{n} \left[c_i j_i + \frac{1}{2} j_i^2 \right]$$

$$j_l = \frac{i + c_k - c_l}{2}, \text{ if } i > j_c$$

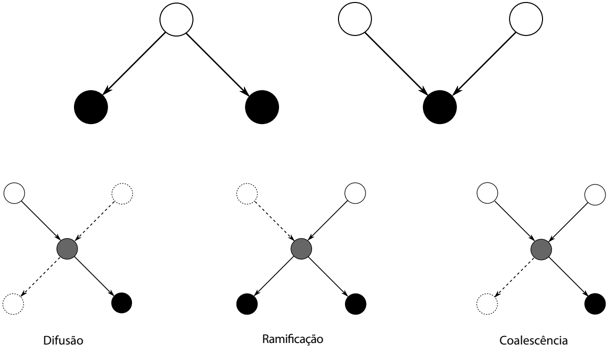
$$j_{l,k} = 0, i, \text{ if } i \leqslant j_c$$

$j_c \equiv c_k - c_l$, e i é a corrente de entrada.

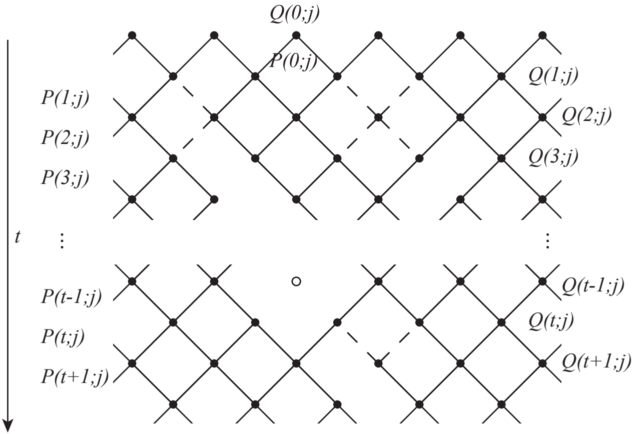


Sumário

Modelo infinito

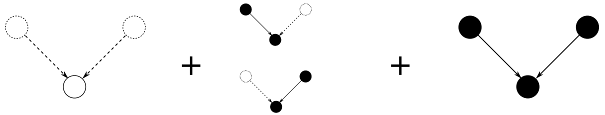


O Modelo



Distribuição de correntes nos vértices dados fluxos nos canais

$Q(\circ) =$



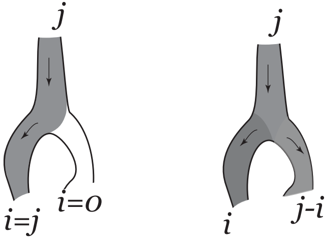
$$\mathcal{Q}(t;i) = B^2(t)\delta(i) + 2B(t)P(t;i) + \int_0^i du P(t;u)P(t;i-u)$$

$$Q(t;i) = 2B(t)P(t;i) + \int_0^i du P(t;u)P(t;i-u)$$

$$\int_0^\infty Q(t;i) = 1 - B^2(t)$$

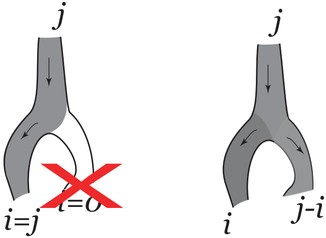
$$\langle j \rangle_Q = 2 \langle j \rangle_P$$

Distribuição de correntes, ligações e vértices



$$F(i|j,j_c) = \left[\frac{1}{2}\delta(i-j) + \frac{1}{2}\delta(i)\right]\Theta(j_c - j) + $$
$$+ \left[\frac{1}{2}\delta\left(i - \frac{j-j_c}{2}\right) + \frac{1}{2}\delta\left(i - \frac{j+j_c}{2}\right)\right]\Theta(j - j_c)$$

Distribuição de correntes, ligações e vértices



$$F^*(i|j,j_c) = \left[\frac{1}{2}\delta(i-j) + \qquad\qquad \right]\Theta(j_c - j) + \\ + \left[\frac{1}{2}\delta\left(i - \frac{j-j_c}{2}\right) + \frac{1}{2}\delta\left(i - \frac{j+j_c}{2}\right)\right]\Theta(j-j_c)$$

Fração de canais vazios

$$P(t+1;i) = 2\int_{S\in\{c'>c''\}}\int_0^\infty dc'dc''djF^*(i|j,j_c)p(c',c'')Q(t;j)$$

$$\int_0^\infty djP(t;j) = 1 - B(t)$$

logo

$$B(t+1) = [B(t)]^2 + \frac{1}{2}\int_0^\infty dj\mathcal{P}_c(j)Q(t;j)$$

onde

$$\mathcal{P}_c(i) \equiv p(j \geqslant i) = 2\int_i^\infty p_c(j)dj,$$

e

$$p_c(i) = \left[p(c') * p(-c'')\right](i),$$

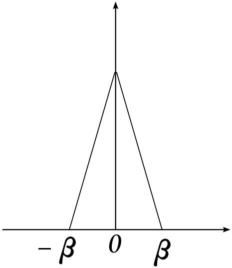
Equações de recorrência

$$
\begin{aligned}
Q(t;i) &= 2B(t)P(t;i) + \int_0^i du P(t;u)P(t;i-u) \\
P(t+1;i) &= \frac{\mathcal{P}_c(i)}{2} Q(i) + 2\int_0^\infty dj p_c(j) Q(2i+j) + \\
&\quad + 2\int_0^i dj p_c(j) Q(2i-j) \\
B(t+1) &= B^2(t) + \frac{1}{2}\int_0^\infty dj \mathcal{P}_c(j) Q(t;j)
\end{aligned}
$$

Distribuição contínua de custos:

Para uma distribuição uniforme de custos:

$$\begin{aligned} p(c) &= \frac{1}{\beta}\Theta(\beta - c) \\ \Rightarrow p_c(j_c) &= \frac{1}{\beta}(\beta - j_c)\Theta(\beta - j_c) \\ \Rightarrow \mathcal{P}_c(j_c) &= \frac{1}{\beta^2}(\beta - j_c)^2\Theta(\beta - j_c) \end{aligned}$$

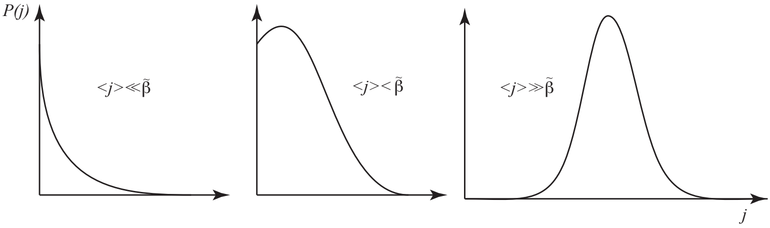


Temos as equações da distribuição de correntes e canais vazios

$$P(t+1;i) = F[\beta, Q(t;i)]$$

$$B(t+1) = B^2(t) + \frac{1}{2\beta^2}\int_0^\beta di(\beta - i)^2 Q(t;i)$$

Distribuições estacionárias



Sumário

$$
\begin{aligned}
K(z) &= 2B\Pi(z) + \Pi(z)^2 \\
\Pi(z) &= \left[\frac{4}{\beta z}\sinh\left(\frac{\beta z}{4}\right)\Pi\left(\frac{z}{2}\right)\right]^2 \\
\Rightarrow \Pi^{(n)}(z) &= \Pi^{(0)}\left(\frac{z}{2^n}\right)^{2^n}\prod_{k=1}^{n}\left[\frac{2^{k+1}}{\beta z}\sinh\left(\frac{\beta z}{2^{k+1}}\right)\right]^{2^k}.
\end{aligned}
$$

Solução estacionária

$$
P(j;\langle j\rangle,\beta) = \frac{1}{\pi\beta}\int_0^\infty dz\cos\left((\langle j\rangle - j)\frac{z}{\beta}\right)\prod_{k=1}^{\infty}\left[\frac{2^{k+1}}{z}\sin\left(\frac{z}{2^{k+1}}\right)\right]^{2^k}
$$

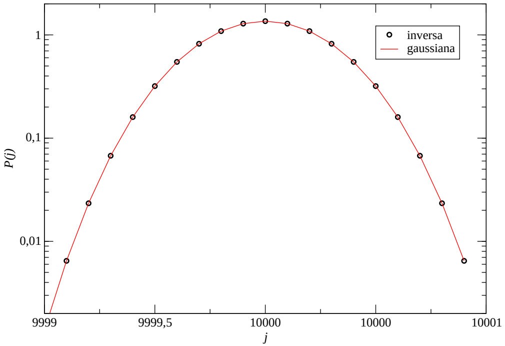


Figure: $\langle j \rangle = 10^4, \beta = 1.\ \sigma^2 = 1/12.$

Sumário

1. Aproximação de campo médio para otimização de fluxos numa rede regular direcionada
   - Motivação
   - Modelo finito
   - Otimização local de fluxos
   - Modelo infinito
   - Limite de grandes correntes
   - Limite de pequenas correntes

Fração de canais usados

$$B = B^2 + \frac{1}{2\beta^2}\int_0^\beta dj(\beta - j)^2 Q(j)$$

$$B^2 - 2B + 1 - 4\frac{\langle j\rangle}{\beta} + \frac{\mu_2^{(Q)}}{\beta^2} = 0$$

$$\Rightarrow B = 1 - \frac{1}{2}\sqrt{\cancel{4} - 4\left(\cancel{1} - 4\frac{\langle j\rangle}{\beta} + \frac{\mu_2^{(Q)}}{\beta^2}\right)}$$

No limite de pequenas correntes $\langle j\rangle \ll \beta$

$$B(\langle j\rangle) = 1 - 2\sqrt{\frac{\langle j\rangle}{\beta}}$$

Limite de pequenas correntes

$$\begin{aligned} P(0) &= \frac{\beta^2}{2\beta^2} Q(0) + \frac{2}{\beta^2} \int_0^\beta dj (\beta - j) Q(j) \\ &= \frac{2}{\beta}(1+B) - \frac{4}{\beta^2} \frac{\langle j \rangle}{1-B} \end{aligned}$$

$$P(j) = 4e^{-\frac{2}{\sqrt{\langle j \rangle}} j}$$

Limite de pequenas correntes

$$
\begin{aligned}
\langle j^n \rangle_P &= F_n[\langle j^n \rangle_Q , \langle j^{n+1} \rangle_Q , \langle j^{n+2} \rangle_Q] \\
\langle j^n \rangle_Q &= G_n[\langle j^1 \rangle_P , ..., \langle j^n \rangle_P]
\end{aligned}
$$

$$
\lim_{\langle j \rangle \to 0} \frac{\langle j^n \rangle (t+1)}{\langle j^n \rangle (t)} = 1
$$

Distribuições estáveis

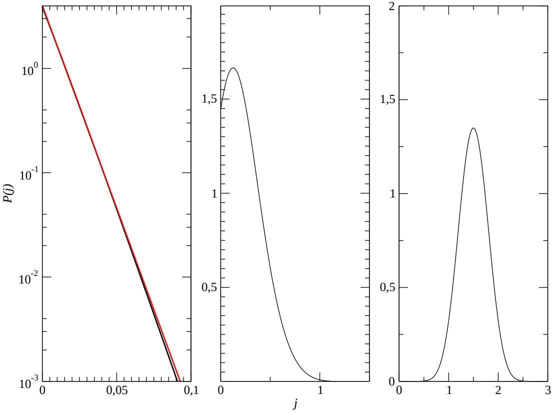
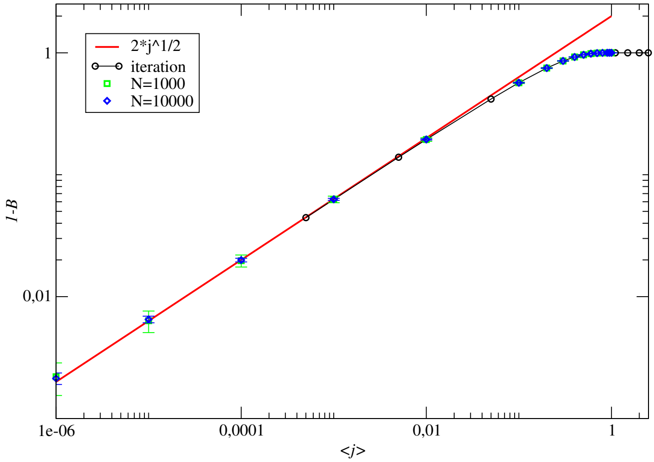


Figure: $\langle j \rangle = 5 \cdot 10^{-4},\ 10^{-2},\ \text{e}\ 5 \cdot 10^{-1}, \beta = 1.$

Fração de canais ativos



Flutuações

Processo de ramificação

$$\mathcal{P}_{bran.}(j \geqslant j_c) = \int_0^\infty dj_c p(j_c) \int_{j_c}^\infty dj P(j)$$

$$\begin{aligned} \mathcal{P}_{bran.}(j \geqslant j_c) &= \int_0^\infty dj_c e^{-j_c} \int_{j_c}^\infty dj A e^{-\frac{\alpha j}{\sqrt{\langle j \rangle}}} \\ &\simeq \frac{\langle j \rangle}{1 + \sqrt{\langle j \rangle}} \sim \langle j \rangle \,. \end{aligned}$$

Flutuações

Processo de coalescência

$$\begin{aligned}\mathcal{P}_{mer.}(j' + j'' \leqslant j_c) &= \int_0^\infty dj_c e^{-j_c} \int_0^{j_c} dj \int_0^j du P(u) P(j-u) \\ &\simeq \frac{\langle j \rangle}{(1+\sqrt{\langle j \rangle})^2} \sim \langle j \rangle \,.\end{aligned}$$

Conclusões

Ocupação média

$$\langle n \rangle \sim \gamma \rightarrow \left(\frac{\langle j \rangle}{\beta} \right)^{\frac{1}{2}} \tag{1}$$

Dupla ocupação

$$\langle n_i n_{i+1} \rangle \sim \gamma^2 \rightarrow \frac{\langle j \rangle}{\beta} \tag{2}$$

Para grandes fluxos as flutuações dependem apenas da largura da distribuição de custos

$$\sigma \rightarrow f(\beta) \tag{3}$$

Obrigado.