In search of lost symmetries

Maria Clara Nucci

University of Perugia & INFN-Perugia, Italy

VI CICLO DE SEMINÁRIOS DE FÍSICA
COLEGIADO DE FÍSICA
Universidade Estadual do Sudoeste da Bahia
December 3, 2020, Itapetinga, BA

Outline

- Linearization of (maximally) superintegrable systems by means of hidden Lie symmetries.

Outline

- Linearization of (maximally) superintegrable systems by means of hidden Lie symmetries.

Outline

- Linearization of (maximally) superintegrable systems by means of hidden Lie symmetries.

- Is Mechanics a branch of mathematical analysis?
  The role of the Jacobi last multiplier: its connection with first integrals, Lagrangians, and Lie symmetries.

Outline

- Linearization of (maximally) superintegrable systems by means of hidden Lie symmetries.

- Is Mechanics a branch of mathematical analysis?
  The role of the Jacobi last multiplier: its connection with first integrals, Lagrangians, and Lie symmetries.

- Quantization through the conservation of Noether symmetries.

Superintegrability and hidden linearity I

The Hamiltonian

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + \frac{\alpha u_2}{u_1^{2/3}}$$

yields a superintegrable (maximally integrable) system

$$\dot{u}_1 = p_1, \qquad \dot{u}_2 = p_2, \qquad \dot{p}_1 = \frac{2\alpha u_2}{3u_1^{5/3}}, \qquad \dot{p}_2 = -\frac{\alpha}{u_1^{2/3}}$$

since there exist two independent integrals of motion [Post and Winternitz, J.Phys.A.,2011], i.e.:

$$
\begin{aligned}
I_1 &= 3p_1^2 p_2 + 2p_2^3 + 9\alpha u_1^{1/3} p_1 + 6\alpha u_2 p_2/(u_1^{2/3}), \\
I_2 &= p_1^4 + 4\alpha u_2 p_1^2/(u_1^{2/3}) - 12 u_1^{1/3} \alpha p_1 p_2 - 2\alpha^2(9u_1^2 - 2u_2^2)/(u_1^{4/3}).
\end{aligned}
$$

Superintegrability and hidden linearity I

The Hamiltonian

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + \frac{\alpha u_2}{u_1^{2/3}}$$

yields a superintegrable (maximally integrable) system

$$\dot{u}_1 = p_1, \qquad \dot{u}_2 = p_2, \qquad \dot{p}_1 = \frac{2\alpha u_2}{3 u_1^{5/3}}, \qquad \dot{p}_2 = -\frac{\alpha}{u_1^{2/3}}$$

since there exist two independent integrals of motion [Post and Winternitz, J.Phys.A.,2011], i.e.:

$$
\begin{aligned}
I_1 &= 3p_1^2 p_2 + 2p_2^3 + 9\alpha u_1^{1/3} p_1 + 6\alpha u_2 p_2/(u_1^{2/3}), \\
I_2 &= p_1^4 + 4\alpha u_2 p_1^2/(u_1^{2/3}) - 12 u_1^{1/3} \alpha p_1 p_2 - 2\alpha^2(9u_1^2 - 2u_2^2)/(u_1^{4/3}).
\end{aligned}
$$

In MCN & Post, JPhysA, 2012, we have determined the hidden linearity of the Lagrangian equations.

The corresponding Lagrangian equations are

$$\ddot{u}_1 = \frac{2\alpha u_2}{3u_1^{5/3}}, \qquad \ddot{u}_2 = -\frac{\alpha}{u_1^{2/3}}.$$

They admits a 2-dim Lie symmetry algebra A_2 generated by

$$\Gamma_1 = \partial_t, \qquad \Gamma_2 = \frac{5}{6}t\partial_t + u_1\partial_{u_1} + u_2\partial_{u_2}.$$

The corresponding Lagrangian equations are

$$\ddot{u}_1 = \frac{2\alpha u_2}{3u_1^{5/3}}, \qquad \ddot{u}_2 = -\frac{\alpha}{u_1^{2/3}}.$$

They admits a 2-dim Lie symmetry algebra A_2 generated by

$$\Gamma_1 = \partial_t, \qquad \Gamma_2 = \frac{5}{6}t\partial_t + u_1\partial_{u_1} + u_2\partial_{u_2}.$$

Since system

$$\dot{u}_1 = p_1, \qquad \dot{u}_2 = p_2, \qquad \dot{p}_1 = \frac{2\alpha u_2}{3u_1^{5/3}}, \qquad \dot{p}_2 = -\frac{\alpha}{u_1^{2/3}}.$$

is autonomous we can choose one of the dependent variables as new independent variable, namely $u_1 = y$. Then

$$u_2' = \frac{p_2}{p_1}, \qquad p_1' = \frac{2\alpha u_2}{3y^{5/3}p_1}, \qquad p_2' = -\frac{\alpha}{y^{2/3}p_1},$$

Eliminating one of the dependent variables, e.g. from the second equation $u_2 = (3y^{5/3}p_1 p_1')/(2\alpha)$ and then the system becomes:

$$p_1'' = \frac{2y^{1/3}\alpha p_2 - 5yp_1^2 p_1' - 3y^2 p_1 p_1'^2}{3y^2 p_1^2}, \qquad p_2' = -\frac{\alpha}{y^{2/3}p_1}. \qquad (1)$$

It admits a 10-dim Lie symmetry algebra isomorphic to the de Sitter algebra $o(3,2)$ generated by:

$$
\begin{aligned}
X_1 &= -\frac{1}{972\alpha^2 y^{2/3}} \Big((3yp_2^8 - 324\alpha^2 y^{5/3}p_2^4 + 8748\alpha^4 y^{7/3} - 972\alpha^2 y^{5/3}p_1^2 p_2^2)\partial_y \\
&\quad + (1944\alpha^3 yp_2^2 p_1 - 108\alpha^2 y^{2/3}p_2^5 + 5832\alpha^4 y^{4/3}p_2)\partial_{p_2} + (2916\alpha^4 y^{4/3}p_1 - p_1 p_2^8 \\
&\quad + 1296\alpha^3 yp_2^3 - 24\alpha y^{1/3}p_2^7 + 432\alpha^2 y^{2/3}p_1 p_2^4 + 324\alpha^2 y^{2/3}p_1^3 p_2^2 + 1944\alpha^3 p_2 p_1^2 y)\partial_{p_1} \Big), \\[4pt]
X_2 &= \frac{1}{486\alpha^2 y^{2/3}} \Big((162\alpha^2 y^{5/3}p_2^3 - 3yp_2^7 + 486\alpha^2 y^{5/3}p_2 p_1^2)\partial_y \\
&\quad + (81\alpha^2 y^{2/3}p_2^4 - 1458\alpha^4 y^{4/3} - 972\alpha^3 yp_1 p_2)\partial_{p_2} + (21\alpha y^{1/3}p_2^6 - 486\alpha^3 yp_2^2 \\
&\quad - 486\alpha^3 yp_1^2 - 270\alpha^2 y^{2/3}p_1 p_2^3 - 162\alpha^2 y^{2/3}p_2 p_1^3 + p_1 p_2^7)\partial_{p_1} \Big), \\[4pt]
X_3 &= \frac{1}{243\alpha^2 y^{2/3}} \Big((243\alpha^2 y^{5/3}p_1^2 - 3yp_2^6)\partial_y + (54\alpha^2 y^{2/3}p_2^3 - 486\alpha^3 yp_1)\partial_{p_2} \\
&\quad + (p_1 p_2^6 - 81\alpha^2 y^{2/3}p_1^3 + 18\alpha y^{1/3}p_2^5 - 162\alpha^2 y^{2/3}p_1 p_2^3)\partial_{p_1} \Big), \\[4pt]
X_4 &= \frac{1}{972\alpha^2 y^{2/3}} \Big((3yp_2^4 + 162y^{5/3}\alpha^2)\partial_y + (54\alpha^2 y^{2/3}p_1 - 12y^{1/3}\alpha p_2^3 - p_1 p_2^4)\partial_{p_1} \Big), \\[4pt]
X_5 &= \frac{1}{27\alpha^2 y^{2/3}} \Big(-3yp_2^3 \partial_y + 27\alpha^2 y^{2/3}\partial_{p_2} + (9p_2^2 y^{1/3}\alpha + p_1 p_2^3)\partial_{p_1} \Big)
\end{aligned}
$$

$$
\begin{aligned}
X_6 &= \frac{1}{36\alpha^2 y^{2/3}} \left((54y^{5/3}\alpha^2 - 3yp_2^4)\partial_y + 36y^{2/3}\alpha^2 p_2 \partial_{p_2} \right. \\
&\quad \left. + (p_2^4 p_1 - 18y^{2/3}\alpha^2 p_1 + 12y^{1/3}\alpha p_2^3)\partial_{p_1} \right), \\
X_7 &= \frac{1}{54\alpha^2 y^{2/3}} \left((162y^{5/3}\alpha^2 p_2 - 3yp_2^5)\partial_y + 54p_2^2 y^{2/3}\alpha^2 \partial_{p_2} \right. \\
&\quad \left. + (p_1 p_2^5 + 15y^{1/3}\alpha p_2^4 - 162\alpha^3 y - 54y^{2/3}\alpha^2 p_1 p_2)\partial_{p_1} \right), \\
X_8 &= \frac{1}{3y^{2/3}}(3y\partial_y - p_1\partial_{p_1}), \\
X_9 &= \frac{1}{3y^{2/3}} \left(3yp_2\partial_y - (3y^{1/3}\alpha + p_1 p_2)\partial_{p_1} \right), \\
X_{10} &= \frac{p_2}{3y^{2/3}} \left(3yp_2\partial_y - (6y^{1/3}\alpha + p_1 p_2)\partial_{p_1} \right),
\end{aligned}
$$

which implies that system (1) is

$$\begin{aligned}
X_6 &= \frac{1}{36\alpha^2 y^{2/3}} \left((54y^{5/3}\alpha^2 - 3yp_2^4)\partial_y + 36y^{2/3}\alpha^2 p_2 \partial_{p_2} \right. \\
&\quad \left. + (p_2^4 p_1 - 18y^{2/3}\alpha^2 p_1 + 12y^{1/3}\alpha p_2^3)\partial_{p_1} \right), \\
X_7 &= \frac{1}{54\alpha^2 y^{2/3}} \left((162y^{5/3}\alpha^2 p_2 - 3yp_2^5)\partial_y + 54p_2^2 y^{2/3}\alpha^2 \partial_{p_2} \right. \\
&\quad \left. + (p_1 p_2^5 + 15y^{1/3}\alpha p_2^4 - 162\alpha^3 y - 54y^{2/3}\alpha^2 p_1 p_2)\partial_{p_1} \right), \\
X_8 &= \frac{1}{3y^{2/3}}(3y\partial_y - p_1\partial_{p_1}), \\
X_9 &= \frac{1}{3y^{2/3}} \left(3yp_2\partial_y - (3y^{1/3}\alpha + p_1 p_2)\partial_{p_1} \right), \\
X_{10} &= \frac{p_2}{3y^{2/3}} \left(3yp_2\partial_y - (6y^{1/3}\alpha + p_1 p_2)\partial_{p_1} \right),
\end{aligned}$$

which implies that system (1) is linearizable!!!

Here is system (2) again:

$$p_1'' = \frac{2y^{1/3}\alpha p_2 - 5yp_1^2 p_1' - 3y^2 p_1 p_1'^2}{3y^2 p_1^2}, \qquad p_2' = -\frac{\alpha}{y^{2/3}p_1}.$$

From the second equation $p_1 = -\alpha/(y^{2/3}p_2')$ yields

$$p_2''' = \frac{1}{9\alpha^2 p_2' y^2} \left(6y^{1/3} p_2 p_2'^5 y^2 + 4\alpha^2 p_2'^2 + 9\alpha^2 p_2' p_2'' y + 27\alpha^2 p_2''^2 y^2 \right).$$

This equation admits a 7-dim Lie symmetry algebra generated by

$$Y_1 = -\frac{y^{1/3}p_2^3}{9\alpha^2}\partial_y + \partial_{p_2}, \;\; Y_2 = \frac{-y^{1/3}p_2^5 + 54\alpha^2 yp_2}{54\alpha^2}\partial_y + \frac{p_2^2}{3}\partial_{p_2}, \;\; Y_3 = \frac{y^{1/3}p_2^4 + 54\alpha^2 y}{54\alpha^2}\partial_y,$$

$$Y_4 = -\frac{y^{1/3}p_2^4}{9\alpha^2}\partial_y + p_2\partial_{p_2}, \;\; Y_5 = y^{1/3}\partial_y, \;\; Y_6 = y^{1/3}p_2\partial_y, \; Y_7 = y^{1/3}p_2^2\partial_y.$$

Thus it is linearizable (10-dim Lie algebra are its contact symm.)

$\tilde{y} = p_2, \; \tilde{p}_2 = \frac{3}{2}y^{2/3} + \frac{1}{36\alpha^2}p_2^4 \quad \Rightarrow$

Here is system (2) again:

$$p_1'' = \frac{2y^{1/3}\alpha p_2 - 5yp_1^2 p_1' - 3y^2 p_1 p_1'^2}{3y^2 p_1^2}, \qquad p_2' = -\frac{\alpha}{y^{2/3} p_1}.$$

From the second equation $p_1 = -\alpha/(y^{2/3} p_2')$ yields

$$p_2''' = \frac{1}{9\alpha^2 p_2' y^2} \left(6y^{1/3} p_2 p_2'^5 y^2 + 4\alpha^2 p_2'^2 + 9\alpha^2 p_2' p_2'' y + 27\alpha^2 p_2''^2 y^2 \right).$$

This equation admits a 7-dim Lie symmetry algebra generated by

$$Y_1 = -\frac{y^{1/3} p_2^3}{9\alpha^2} \partial_y + \partial_{p_2}, \quad Y_2 = \frac{-y^{1/3} p_2^5 + 54\alpha^2 y p_2}{54\alpha^2} \partial_y + \frac{p_2^2}{3} \partial_{p_2}, \quad Y_3 = \frac{y^{1/3} p_2^4 + 54\alpha^2 y}{54\alpha^2} \partial_y,$$

$$Y_4 = -\frac{y^{1/3} p_2^4}{9\alpha^2} \partial_y + p_2 \partial_{p_2}, \quad Y_5 = y^{1/3} \partial_y, \quad Y_6 = y^{1/3} p_2 \partial_y, \quad Y_7 = y^{1/3} p_2^2 \partial_y.$$

Thus it is linearizable (10-dim Lie algebra are its contact symm.)

$$\tilde{y} = p_2, \ \tilde{p}_2 = \frac{3}{2} y^{2/3} + \frac{1}{36\alpha^2} p_2^4 \quad \Rightarrow \quad \frac{\mathrm{d}^3 \tilde{p}_2}{\mathrm{d}\tilde{y}^3} = 0$$

Since $y = u_1$ and $u_2 = (3y^{5/3}p_1p_1')/(2\alpha)$ then the 10-dim Lie symmetry algebra is obtained:

$$
\begin{aligned}
\Gamma_1 &= V_1\partial_t + \frac{1}{972\alpha^3 u_1^{4/3}}\left((324u_1^{7/3}\alpha^3 p_2^4 - 3u_1^{5/3}\alpha p_2^8 - 8748u_1^3\alpha^5 + 972u_1^{7/3}\alpha^3 p_2^2 p_1^2)\partial_{u_1}\right.\\
&\quad +(5832u_1^{8/3}p_2^2\alpha^4 - 252\alpha^2 u_1^2 p_2^6 - 5832\alpha^5 u_1^2 u_2 - u_1^{4/3}p_2^8 p_1^2 - 24\alpha u_1^{5/3}p_2^7 p_1\\
&\quad -2\alpha u_1^{2/3}p_2^8 u_2 - 3888u_1^{5/3}p_2\alpha^4 u_2 p_1 - 1296u_1^{2/3}p_2^2\alpha^4 u_2^2 - 648u_1^{4/3}p_2^2\alpha^3 u_2 p_1^2\\
&\quad -648u_1^{4/3}p_2^4\alpha^3 u_2 + 1944u_1^{7/3}p_2^3\alpha^3 p_1)\partial_{u_2}\\
&\quad +(24u_1\alpha^2 p_2^7 - 2916u_1^2\alpha^5 p_1 + u_1^{2/3}\alpha p_2^8 p_1 - 1296u_1^{5/3}\alpha^4 p_2^3\\
&\quad -432u_1^{4/3}\alpha^3 p_2^4 p_1 - 324u_1^{4/3}\alpha^3 p_2^2 p_1^3 - 1944u_1^{5/3}\alpha^4 p_2 p_1^2)\partial_{p_1}\\
&\quad \left.+(108p_2^5 u_1^{4/3}\alpha^3 - 5832p_2 u_1^2\alpha^5 - 1944p_2^2 u_1^{5/3}\alpha^4 p_1)\partial_{p_2}\right),\\[6pt]
\Gamma_2 &= V_2\partial_t + \frac{1}{486\alpha^3 u_1^{4/3}}\left((-3u_1^{5/3}p_2^7\alpha + 162u_1^{7/3}p_2^3\alpha^3 + 486u_1^{7/3}p_2\alpha^3 p_1^2)\partial_{u_1}\right.\\
&\quad +(972u_1^{7/3}p_2^2\alpha^3 p_1 - 21\alpha u_1^{5/3}p_2^6 p_1 - 2\alpha u_1^{2/3}p_2^7 u_2 + 1458u_1^{8/3}\alpha^4 p_2\\
&\quad -972u_1^{5/3}\alpha^4 u_2 p_1 - 648u_1^{2/3}\alpha^4 u_2^2 p_2 - 189p_2^5\alpha^2 u_1^2 - u_1^{4/3}p_2^7 p_1^2\\
&\quad -324u_1^{4/3}p_2\alpha^3 u_2 p_1^2 - 432u_1^{4/3}p_2^3\alpha^3 u_2)\partial_{u_2}\\
&\quad +(u_1^{2/3}\alpha p_2^7 p_1 + 21u_1\alpha^2 p_2^6 - 486u_1^{5/3}\alpha^4 p_2^2 - 486u_1^{5/3}\alpha^4 p_1^2 - 270u_1^{4/3}\alpha^3 p_2^3 p_1\\
&\quad \left.-162u_1^{4/3}\alpha^3 p_2 p_1^3)\partial_{p_1} + (81u_1^{4/3}\alpha^3 p_2^4 - 1458u_1^2\alpha^5 - 972u_1^{5/3}\alpha^4 p_2 p_1)\partial_{p_2}\right),
\end{aligned}
$$

$$\Gamma_3 = V_3\partial_t + \frac{1}{243\alpha^3 u_1^{4/3}}\left((243u_1^{7/3}\alpha^3 p_1^2 - 3u_1^{5/3}\alpha p_2^6)\partial_{u_1}\right.$$

$$-(18\alpha u_1^{5/3}p_2^5 p_1 + 2\alpha u_1^{2/3}p_2^6 u_2 + 324u_1^{2/3}u_2^2\alpha^4 + 135p_2^4\alpha^2 u_1^2 + u_1^{4/3}p_2^6 p_1^2$$

$$+162u_1^{4/3}\alpha^3 u_2 p_1^2 + 324u_1^{4/3}\alpha^3 u_2 p_2^2 - 486u_1^{7/3}\alpha^3 p_1 p_2)\partial_{u_2}$$

$$+(u_1^{2/3}\alpha p_2^6 p_1 - 81u_1^{4/3}\alpha^3 p_1^3 + 18u_1\alpha^2 p_2^5 - 162u_1^{4/3}\alpha^3 p_2^2 p_1)\partial_{p_1}$$

$$+(54u_1^{4/3}\alpha^3 p_2^3 - 486u_1^{5/3}\alpha^4 p_1)\partial_{p_2}\right),$$

$$\Gamma_4 = V_4\partial_t + \frac{1}{972\alpha^3 u_1^{4/3}}\left((3u_1^{5/3}\alpha p_2^4 + 162u_1^{7/3}\alpha^3)\partial_{u_1}\right.$$

$$+(12\alpha u_1^{5/3}p_2^3 p_1 + 2\alpha u_1^{2/3}p_2^4 u_2 + 54p_2^2\alpha^2 u_1^2 + u_1^{4/3}p_2^4 p_1^2 + 216u_1^{4/3}\alpha^3 u_2)\partial_{u_2}$$

$$+(54u_1^{4/3}\alpha^3 p_1 - u_1^{2/3}\alpha p_2^4 p_1 - 12u_1\alpha^2 p_2^3)\partial_{p_1}\right),$$

$$\Gamma_5 = V_5\partial_t - \frac{u_1^{1/3}p_2^3}{9\alpha^2}\partial_{u_1} - \frac{1}{27\alpha^3 u_1^{4/3}}\left((9p_2^2\alpha u_1^{5/3}p_1 + 2p_2^3\alpha u_1^{2/3}u_2\right.$$

$$+27p_2\alpha^2 u_1^2 + p_2^3 u_1^{4/3}p_1^2)\partial_{u_2} - (p_2^3 u_1^{2/3}\alpha p_1 + 9p_2^2 u_1\alpha^2)\partial_{p_1}\right) + \partial_{p_2},$$

$$\Gamma_6 = V_6\partial_t + \frac{1}{36\alpha^3 u_1^{4/3}}\left((54u_1^{7/3}\alpha^3 - 3u_1^{5/3}\alpha p_2^4)\partial_{u_1}\right.$$

$$-(12\alpha u_1^{5/3}p_2^3 p_1 + 2\alpha u_1^{2/3}p_2^4 u_2 + 54p_2^2\alpha^2 u_1^2 + u_1^{4/3}p_2^4 p_1^2)\partial_{p_2}$$

$$+(u_1^{2/3}\alpha p_2^4 p_1 - 18u_1^{4/3}\alpha^3 p_1 + 12u_1\alpha^2 p_2^3)\partial_{p_1}\right) + p_2\partial_{p_2},$$

$$\Gamma_7 = V_7\partial_t + \frac{1}{54\alpha^3 u_1^{4/3}}\left((162u_1^{7/3}p_2\alpha^3 - 3u_1^{5/3}p_2^5\alpha)\partial_{u_1}\right.$$

$$-(15p_2^4\alpha u_1^{5/3}p_1 + 2p_2^5\alpha u_1^{2/3}u_2 + 90p_2^3\alpha^2 u_1^2 + u_1^{4/3}p_2^5 p_1^2)\partial_{u_2}$$

$$+(u_1^{2/3}\alpha p_2^5 p_1 + 15u_1\alpha^2 p_2^4 - 162u_1^{5/3}\alpha^4 - 54u_1^{4/3}\alpha^3 p_2 p_1)\partial_{p_1}\right) + p_2^2\partial_{p_2},$$

$$\Gamma_8 = V_8 \partial_t + \frac{1}{3\alpha u_1^{2/3}}\left(3\alpha u_1 \partial_{u_1} + (2u_2\alpha + u_1^{2/3}p_1^2)\partial_{u_2} - \alpha p_1 \partial_{p_1}\right),$$

$$\Gamma_9 = V_9 \partial_t + \frac{1}{3\alpha u_1^{2/3}}\left(3\alpha u_1 p_2 \partial_{u_1} + (3\alpha p_1 u_1 + 2\alpha u_2 p_2 + u_1^{2/3}p_2 p_1^2)\partial_{u_2}\right.$$
$$\left. -(\alpha p_2 p_1 + 3u_1^{1/3}\alpha^2)\partial_{p_1}\right),$$

$$\Gamma_{10} = V_{10}\partial_t + u_1^{1/3}p_2^2 \partial_{u_1} + \frac{1}{3\alpha u_1^{4/3}}\left((6\alpha p_1 p_2 u_1^{5/3} + 2\alpha p_2^2 u_1^{2/3}u_2\right.$$
$$\left. + 9\alpha^2 u_1^2 + p_1^2 p_2^2 u_1^{4/3})\partial_{u_2} - \alpha p_2(6\alpha u_1 + p_1 p_2 u_1^{2/3})\partial_{p_1}\right).$$

Each $V_k = V_k(t, u_1, u_2, p_1, p_2)$, $(k = 1, \ldots, 10)$, satisfies:

$$486 u_1^{5/3}\alpha^2 V_y + 486 u_1^{5/3}\alpha^2 p_1 V_{u_1} + 486 u_1^{5/3}\alpha^2 p_2 V_{u_2} + 324\alpha^3 u_2 V_{p_1} - 486\alpha^3 u_1 V_{p_2}$$
$$+ c_1(u_1 p_2^8 - 648 u_1\alpha^3 u_2 p_2^2 + 5832 u_1^{7/3}\alpha^4 - 648 u_1^{5/3}\alpha^2 p_2^2 p_1^2 - 378 u_1^{5/3}\alpha^2 p_2^4)$$
$$- c_2(648 u_1\alpha^3 u_2 p_2 + 2u_1 p_2^7 + 648 u_1^{5/3}\alpha^2 p_2 p_1^2 + 432 u_1^{5/3}\alpha^2 p_2^3)$$
$$+ c_3(4u_1 p_2^6 - 648 u_1\alpha^3 u_2 - 648 u_1^{5/3}\alpha^2 p_1^2 - 324 u_1^{5/3}\alpha^2 p_2^2)$$
$$- c_4(u_1 p_2^4 + 54 u_1^{5/3}\alpha^2) + 36 u_1 p_2^3 c_5 + c_6(27 u_1 p_2^2 - 972 u_1^{5/3}\alpha^2)$$
$$+ c_7(18 u_1 p_2^5 - 1944 u_1^{1/3}\alpha^2 p_2) - 324 u_1\alpha^2 c_8 - 324\alpha^2 p_2 u_1 c_9 - 324 u_1\alpha^2 p_2^2 c_{10} = 0$$

with $c_k = 1$ and all the other $c_{j\neq k} = 0$, $(j = 1, \ldots, 10)$.

Superintegrability and hidden linearity II

In *Friš, Mandrosov, Smorodinsky, Uhlìř and Winternitz, Phys.Lett. A, 1965* the following Hamiltonians were considered

$$H = \tfrac{1}{2}\left(p_1^2 + p_2^2\right) + V(x_1, x_2), \quad H = \tfrac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + V(r, \varphi),$$

and it was proven that only four independent potentials exist such that two quadratical integrals of motion exist.

Superintegrability and hidden linearity II

In *Friš, Mandrosov, Smorodinsky, Uhlìř and Winternitz, Phys.Lett. A, 1965* the following Hamiltonians were considered

$$H = \tfrac{1}{2}\left(p_1^2 + p_2^2\right) + V(x_1, x_2), \quad H = \tfrac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + V(r, \varphi),$$

and it was proven that only four independent potentials exist such that two quadratical integrals of motion exist.

In *G.Gubbiotti & MCN, J.Math.Phys., 2017* we have proven that the corresponding Lagrange equations can be linearized through hidden symmetries.

Superintegrability and hidden linearity II

In *Friš, Mandrosov, Smorodinsky, Uhlìř and Winternitz, Phys.Lett. A, 1965* the following Hamiltonians were considered

$$H = \tfrac{1}{2}\left(p_1^2 + p_2^2\right) + V(x_1, x_2), \quad H = \tfrac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + V(r, \varphi),$$

and it was proven that only four independent potentials exist such that two quadratical integrals of motion exist.

In *G.Gubbiotti & MCN, J.Math.Phys., 2017* we have proven that the corresponding Lagrange equations can be linearized through hidden symmetries. N.B. Regardless of separability in the Hamilton-Jacobi equation, as shown in *MCN & S. Post, J.Phys.A, 2012* for the Post-Winternitz system *Post and Winternitz, J.Phys.A, 2011*.

Superintegrability and hidden linearity II

In *Friš, Mandrosov, Smorodinsky, Uhlìř and Winternitz, Phys.Lett. A, 1965* the following Hamiltonians were considered

$$H = \tfrac{1}{2}\left(p_1^2 + p_2^2\right) + V(x_1, x_2), \quad H = \tfrac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + V(r, \varphi),$$

and it was proven that only four independent potentials exist such that two quadratical integrals of motion exist.

In *G.Gubbiotti & MCN, J.Math.Phys., 2017* we have proven that the corresponding Lagrange equations can be linearized through hidden symmetries. N.B. Regardless of separability in the Hamilton-Jacobi equation, as shown in *MCN & S. Post, J.Phys.A, 2012* for the Post-Winternitz system *Post and Winternitz, J.Phys.A, 2011*. Among other cases, we have proven that the superintegrable Tremblay-Turbiner-Winternitz (TTW) equations hide linearizability through symmetries by raising the order.

Superintegrability and hidden linearity II

In *Friš, Mandrosov, Smorodinsky, Uhlìř and Winternitz, Phys.Lett. A, 1965* the following Hamiltonians were considered

$$H = \tfrac{1}{2}\left(p_1^2 + p_2^2\right) + V(x_1, x_2), \quad H = \tfrac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + V(r, \varphi),$$

and it was proven that only four independent potentials exist such that two quadratical integrals of motion exist.

In *G.Gubbiotti & MCN, J.Math.Phys., 2017* we have proven that the corresponding Lagrange equations can be linearized through hidden symmetries. N.B. Regardless of separability in the Hamilton-Jacobi equation, as shown in *MCN & S. Post, J.Phys.A, 2012* for the Post-Winternitz system *Post and Winternitz, J.Phys.A, 2011.* Among other cases, we have proven that the superintegrable Tremblay-Turbiner-Winternitz (TTW) equations hide linearizability through symmetries by raising the order.

Superintegrability and hidden linearity II

In *Friš, Mandrosov, Smorodinsky, Uhlìř and Winternitz, Phys.Lett. A, 1965* the following Hamiltonians were considered

$$H = \tfrac{1}{2}\left(p_1^2 + p_2^2\right) + V(x_1, x_2), \quad H = \tfrac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) + V(r, \varphi),$$

and it was proven that only four independent potentials exist such that two quadratical integrals of motion exist.

In *G.Gubbiotti & MCN, J.Math.Phys., 2017* we have proven that the corresponding Lagrange equations can be linearized through hidden symmetries. N.B. Regardless of separability in the Hamilton-Jacobi equation, as shown in *MCN & S. Post, J.Phys.A,*

CONJECTURE:
ARE ALL SUPERINTEGRABLE SYSTEMS LINEARIZABLE
IN 2-DIMENSIONAL SPACE?

hide linearizability through symmetries by raising the order.

TTW system: raising the order

The potential is

$$V(r, \varphi) = \omega^2 r^2 + \frac{k^2}{r^2}\left(\frac{\beta_1}{\cos^2(k\varphi)} + \frac{\beta_2}{\sin^2(k\varphi)}\right).$$

The Lagrangian equations are:

$$\ddot{r} = -4\omega^2 r + r\dot{\varphi}^2 + \frac{4k^2}{r^3}\left(\frac{\beta_1}{\cos^2(k\varphi)} + \frac{\beta_2}{\sin^2(k\varphi)}\right),$$

$$\ddot{\varphi} = -\frac{2\dot{r}\dot{\varphi}}{r} - \frac{4k^3}{r^4}\left(\frac{\beta_1 \sin(k\varphi)}{\cos^3(k\varphi)} - \frac{\beta_2 \cos(k\varphi)}{\sin^3(k\varphi)}\right).$$

They admit a three-dimensional Lie symmetry algebra generated by:

$$\Sigma_1 = \partial_t, \quad \Sigma_2 = \cos(4\omega t)\partial_t - 2\omega \sin(4\omega t) r \partial_r,$$

$$\Sigma_3 = \sin(4\omega t)\partial_t + 2\omega \cos(4\omega t) r \partial_r,$$

By solving the Lagrangian equations with respect to β_1 and β_2, and then taking the derivative with respect to t, the following system of two equations of third order is obtained:

By solving the Lagrangian equations with respect to β_1 and β_2, and then taking the derivative with respect to t, the following system of two equations of third order is obtained:

$$r\,\dddot{r} + 16\omega^2\dot{r} + 3\dot{r}\ddot{r} = 0, \quad (*)$$

$$\cos(k\varphi)\sin(k\varphi)r^2\,\dddot{\varphi} + 3\cos^2(k\varphi)kr^2\dot{\varphi}\ddot{\varphi}$$
$$+\, 6\cos^2(k\varphi)kr\dot{r}\dot{\varphi}^2 + 8\cos(k\varphi)\sin(k\varphi)\alpha k^2 r^2\dot{\varphi}$$
$$-\, 4\cos(k\varphi)\sin(k\varphi)k^2r^2\dot{\varphi}^3 + 4\cos(k\varphi)\sin(k\varphi)k^2r\ddot{r}\dot{\varphi}$$
$$+\, 6\cos(k\varphi)\sin(k\varphi)r\dot{r}\ddot{\varphi} + 2\cos(k\varphi)\sin(k\varphi)r\ddot{r}\dot{\varphi}$$
$$+\, 6\cos(k\varphi)\sin(k\varphi)\dot{r}^2\dot{\varphi} - 3\sin^2(k\varphi)kr^2\dot{\varphi}\ddot{\varphi} - 6\sin^2(k\varphi)kr\dot{r}\dot{\varphi}^2 = 0. \ (**)$$

By solving the Lagrangian equations with respect to β_1 and β_2, and then taking the derivative with respect to t, the following system of two equations of third order is obtained:

$$r\,\dddot{r} + 16\omega^2\dot{r} + 3\dot{r}\ddot{r} = 0, \quad (*)$$

$$\cos(k\varphi)\sin(k\varphi)r^2\,\dddot{\varphi} + 3\cos^2(k\varphi)kr^2\dot{\varphi}\ddot{\varphi}$$
$$+ 6\cos^2(k\varphi)kr\dot{r}\dot{\varphi}^2 + 8\cos(k\varphi)\sin(k\varphi)\alpha k^2 r^2\dot{\varphi}$$
$$- 4\cos(k\varphi)\sin(k\varphi)k^2 r^2\dot{\varphi}^3 + 4\cos(k\varphi)\sin(k\varphi)k^2 r\ddot{r}\dot{\varphi}$$
$$+ 6\cos(k\varphi)\sin(k\varphi)r\dot{r}\ddot{\varphi} + 2\cos(k\varphi)\sin(k\varphi)r\ddot{r}\dot{\varphi}$$
$$+ 6\cos(k\varphi)\sin(k\varphi)\dot{r}^2\dot{\varphi} - 3\sin^2(k\varphi)kr^2\dot{\varphi}\ddot{\varphi} - 6\sin^2(k\varphi)kr\dot{r}\dot{\varphi}^2 = 0. \ (**)$$

The first equation admits a 7-dim Lie sym algebra generated by:
$X_1 = \partial_t$, $X_2 = \cos(4\omega t)\partial_t - 2\omega\sin(4\omega t)r\partial_r$,
$X_3 = \sin(4\omega t)\partial_t + 2\omega\cos(4\omega t)r\partial_r$, $X_4 = \frac{\cos(4\omega t)}{r}\partial_r$,
$X_5 = \frac{\sin(4\omega t)}{r}\partial_r$, $X_6 = r\partial_r$, $X_7 = \frac{1}{r}\partial_r$, and consequently it is linearizable.

By solving the Lagrangian equations with respect to β_1 and β_2, and then taking the derivative with respect to t, the following system of two equations of third order is obtained:

$$r\,\dddot{r} + 16\omega^2\dot{r} + 3\dot{r}\ddot{r} = 0, \quad (*)$$

$$\cos(k\varphi)\sin(k\varphi)r^2\,\dddot{\varphi} + 3\cos^2(k\varphi)kr^2\dot{\varphi}\ddot{\varphi}$$
$$+ 6\cos^2(k\varphi)kr\dot{r}\dot{\varphi}^2 + 8\cos(k\varphi)\sin(k\varphi)\alpha k^2 r^2\dot{\varphi}$$
$$- 4\cos(k\varphi)\sin(k\varphi)k^2r^2\dot{\varphi}^3 + 4\cos(k\varphi)\sin(k\varphi)k^2 r\ddot{r}\dot{\varphi}$$
$$+ 6\cos(k\varphi)\sin(k\varphi)r\dot{r}\ddot{\varphi} + 2\cos(k\varphi)\sin(k\varphi)r\ddot{r}\dot{\varphi}$$
$$+ 6\cos(k\varphi)\sin(k\varphi)\dot{r}^2\dot{\varphi} - 3\sin^2(k\varphi)kr^2\ddot{\varphi}\ddot{\varphi} - 6\sin^2(k\varphi)kr\dot{r}\dot{\varphi}^2 = 0. \; (**)$$

The first equation admits a 7-dim Lie sym algebra generated by:
$X_1 = \partial_t$, $X_2 = \cos(4\omega t)\partial_t - 2\omega\sin(4\omega t)r\partial_r$,
$X_3 = \sin(4\omega t)\partial_t + 2\omega\cos(4\omega t)r\partial_r$, $X_4 = \frac{\cos(4\omega t)}{r}\partial_r$,
$X_5 = \frac{\sin(4\omega t)}{r}\partial_r$, $X_6 = r\partial_r$, $X_7 = \frac{1}{r}\partial_r$, and consequently it is
linearizable. A 2-dim nonabelian intransitive subalgebra is

$< X_6, X_7 >$, that yields $u = r^2/2$ and thus $\ddot{u} = -16\omega^2\dot{u}$. Finally
$r = \sqrt{a_1 + a_2\cos(4\omega t) + a_3\sin(4\omega t)}$.

Also the second equation $(**)$ is linearizable since it admits a 7-dim Lie symmetry algebra generated by:

$$\Omega = s_1(t)\partial_t + \frac{-\cos^2(k\varphi)s_2(t) + 2ks_3(t)}{2\cos(k\varphi)\sin(k\varphi)k}\partial_\varphi,$$

with s_1, s_2, s_3 that satisfy the following seventh-order linear system:

$$r^2\,\dddot{s}_1 + 4\dot{s}_1\ddot{r}k^2r - 4\dot{s}_1\ddot{r}r + 16\dot{s}_1k^2\omega^2r^2 - 8\ddot{r}\ddot{r}k^2s_1$$
$$+ 8\ddot{r}\dot{r}s_1 - 32\dot{r}k^2\omega^2s_1r + 32\dot{r}\omega^2s_1r = 0,$$

$$r^2\dot{s}_2 - \ddot{s}_1r^2 + 2\dot{s}_1r\dot{r} + 2r\ddot{r}s_1 - 2\dot{r}^2s_1 = 0,$$

$$r^2\,\dddot{s}_3 + 6\ddot{s}_3\dot{r}r + 4\dot{s}_3\ddot{r}k^2r + \dot{s}_3\ddot{r}r + 6\dot{s}_3\dot{r}^2 + 16\dot{s}_3k^2\omega^2r^2 = 0.$$

Also the second equation $(**)$ is linearizable since it admits a 7-dim Lie symmetry algebra generated by:

$$\Omega = s_1(t)\partial_t + \frac{-\cos^2(k\varphi)s_2(t) + 2ks_3(t)}{2\cos(k\varphi)\sin(k\varphi)k}\partial_\varphi,$$

with s_1, s_2, s_3 that satisfy the following seventh-order linear system:

$$r^2\,\dddot{s}_1 + 4\dot{s}_1\ddot{r}k^2r - 4\dot{s}_1\ddot{r}r + 16\dot{s}_1k^2\omega^2r^2 - 8\ddot{r}\dot{r}k^2s_1$$
$$+ 8\ddot{r}\dot{r}s_1 - 32\dot{r}k^2\omega^2s_1r + 32\dot{r}\omega^2s_1r = 0,$$

$$r^2\dot{s}_2 - \ddot{s}_1r^2 + 2\dot{s}_1r\dot{r} + 2r\ddot{r}s_1 - 2\dot{r}^2s_1 = 0,$$

$$r^2\,\dddot{s}_3 + 6\ddot{s}_3\dot{r}r + 4\dot{s}_3\ddot{r}k^2r + \dot{s}_3\ddot{r}r + 6\dot{s}_3\dot{r}^2 + 16\dot{s}_3k^2\omega^2r^2 = 0.$$

Also the second equation $(**)$ is linearizable since it admits a 7-dim Lie symmetry algebra generated by:

$$\Omega = s_1(t)\partial_t + \frac{-\cos^2(k\varphi)s_2(t) + 2ks_3(t)}{2\cos(k\varphi)\sin(k\varphi)k}\partial_\varphi,$$

with s_1, s_2, s_3 that satisfy the following seventh-order linear system:

$$r^2\,\dddot{s}_1 + 4\dot{s}_1\ddot{r}k^2r - 4\dot{s}_1\ddot{r}r + 16\dot{s}_1k^2\omega^2r^2 - 8\ddot{r}\dot{r}k^2s_1$$
$$+ 8\ddot{r}\dot{r}s_1 - 32\dot{r}k^2\omega^2s_1r + 32\dot{r}\omega^2s_1r = 0,$$

$$r^2\dot{s}_2 - \ddot{s}_1r^2 + 2\dot{s}_1r\dot{r} + 2r\ddot{r}s_1 - 2\dot{r}^2s_1 = 0,$$

$$r^2\,\dddot{s}_3 + 6\ddot{s}_3\dot{r}r + 4\dot{s}_3\ddot{r}k^2r + \dot{s}_3\ddot{r}r + 6\dot{s}_3\dot{r}^2 + 16\dot{s}_3k^2\omega^2r^2 = 0.$$

Also the second equation (**) is linearizable since it admits a 7-dim Lie symmetry algebra generated by:

$$\Omega = s_1(t)\partial_t + \frac{-\cos^2(k\varphi)s_2(t) + 2ks_3(t)}{2\cos(k\varphi)\sin(k\varphi)k}\partial_\varphi,$$

with s_1, s_2, s_3 that satisfy the following seventh-order linear system:

$$r^2\,\dddot{s}_1 + 4\dot{s}_1\ddot{r}k^2r - 4\dot{s}_1\ddot{r}r + 16\dot{s}_1k^2\omega^2r^2 - 8\ddot{r}\dot{r}k^2s_1$$
$$+ 8\ddot{r}\dot{r}s_1 - 32\dot{r}k^2\omega^2s_1r + 32\dot{r}\omega^2s_1r = 0,$$

$$r^2\dot{s}_2 - \ddot{s}_1r^2 + 2\dot{s}_1r\dot{r} + 2r\ddot{r}s_1 - 2\dot{r}^2s_1 = 0,$$

$$r^2\,\dddot{s}_3 + 6\ddot{s}_3\dot{r}r + 4\dot{s}_3\ddot{r}k^2r + \dot{s}_3\ddot{r}r + 6\dot{s}_3\dot{r}^2 + 16\dot{s}_3k^2\omega^2r^2 = 0.$$

$< -\frac{1}{2k}\cot(k\varphi)\partial_\varphi, \frac{2}{\sin(2k\varphi)}\partial_\varphi >$ is a 2-dim nonabelian intransitive subalgebra. Then the second equation (**) becomes linear by means of the canonical transformation $v = -\frac{1}{2k}\cos^2(k\varphi)$, i.e.

$$\dddot{v} = -\frac{6\dot{r}}{r}\ddot{v} - \frac{2}{r^2}\left(3\dot{r}^2 + 8k^2\omega^2r^2 + (2k^2+1)r\ddot{r}\right)\dot{v}.$$

More hidden linearity

In *G.Gubbiotti & MCN, 2020* we have determined the hidden linearity of all Darboux spaces of Type I-IV *Kalnins, Kress, Miller, Winternitz,, J.Math.Phys., 2003*, and Bertrand-Perlick *Perlick, Class. Quant. Grav., 1992*, *Riglioni, J.Phys.A, 2013*.

More hidden linearity

In *G.Gubbiotti & MCN, 2020* we have determined the hidden linearity of all Darboux spaces of Type I-IV *Kalnins, Kress, Miller, Winternitz,, J.Math.Phys., 2003*, and Bertrand-Perlick *Perlick, Class. Quant. Grav., 1992*, *Riglioni, J.Phys.A, 2013*. In *Evans, Phys.Rev. A, 1990*, four Hamiltonians were derived. In *MCN & R.Campoamor-Stursberg, 2020* we have proved that they all hide linearity. For example,

More hidden linearity

In *G.Gubbiotti & MCN, 2020* we have determined the hidden linearity of all Darboux spaces of Type I-IV *Kalnins, Kress, Miller, Winternitz,, J.Math.Phys., 2003*, and Bertrand-Perlick *Perlick, Class. Quant. Grav., 1992, Riglioni, J.Phys.A, 2013*. In *Evans, Phys.Rev. A, 1990*, four Hamiltonians were derived. In *MCN & R.Campoamor-Stursberg, 2020* we have proved that they all hide linearity. For example,

$$H_3 = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + \frac{k_1 w_1}{w_2^2 \sqrt{w_1^2 + w_2^2}} + \frac{k_2}{w_2^2} + \frac{k_3}{w_3^2}$$

N.B. Its Hamilton-Jacobi eq is NOT separable in cartesian coordinates

More hidden linearity

In *G.Gubbiotti & MCN, 2020* we have determined the hidden linearity of all Darboux spaces of Type I-IV *Kalnins, Kress, Miller, Winternitz,, J.Math.Phys., 2003*, and Bertrand-Perlick *Perlick, Class. Quant. Grav., 1992*, *Riglioni, J.Phys.A, 2013*. In *Evans, Phys.Rev. A, 1990*, four Hamiltonians were derived. In *MCN & R.Campoamor-Stursberg, 2020* we have proved that they all hide linearity. For example,

$$H_3 = \tfrac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + \frac{k_1 w_1}{w_2^2\sqrt{w_1^2 + w_2^2}} + \frac{k_2}{w_2^2} + \frac{k_3}{w_3^2}$$

N.B. Its Hamilton-Jacobi eq is NOT separable in cartesian coordinates yields the Hamilton equations

$$\dot{w}_1 = p_1, \quad \dot{w}_2 = p_2, \quad \dot{w}_3 = p_3, \quad \dot{p}_1 = -\frac{k_1}{(w_1^2 + w_2^2)^{3/2}},$$

$$\dot{p}_2 = \frac{k_1 w_1(2w_1^2 + 3w_2^2)}{w_2^3(w_1^2 + w_2^2)^{3/2}} + \frac{2k_2}{w_2^3}, \quad \dot{p}_3 = \frac{2k_3}{w_3^3}.$$

Then $p_3 = \dot{w}_3 \implies \ddot{w}_3 = \frac{2k_3}{w_3^3},$

Solving $\ddot{w}_3 = \dfrac{2k_3}{w_3^3}$ with respect to k_3 and deriving once with respect to t, yields:

$$\dddot{w}_3 = -\frac{3\dot{w}_3 \ddot{w}_3}{w_3},$$

which admits a 7-dim Lie symmetry algebra generated by

$$X_1 = t^2\partial_t + tw_3\partial_{w_3}, \ X_2 = t\partial_t, \ X_3 = \partial_t, \ X_4 = w_3\partial_{w_3},$$

$$X_5 = \frac{t^2}{w_3}\partial_{w_3}, \ X_6 = \frac{t}{w_3}\partial_{w_3}, \ X_7 = \frac{1}{w_3}\partial_{w_3},$$

and therefore it is linearizable. In fact, the new dependent variable $u = w_3^2/2$ transforms it into the linear equation

$$\dddot{u} = 0,$$

and thus the general solution is

$$w_3 = \pm\sqrt{A_1 t^2 + A_2 t + \frac{A_2^2 + 8k_3}{4A_1}},$$

with $A_n, (n = 1, 2)$ arbitrary constants.

About the other four equations of the Hamilton H_3 system:

$$\dot{w}_1 = p_1, \quad \dot{w}_2 = p_2, \quad \dot{p}_1 = -\frac{k_1}{(w_1^2 + w_2^2)^{3/2}},$$

$$\dot{p}_2 = \frac{k_1 w_1 (2w_1^2 + 3w_2^2)}{w_2^3 (w_1^2 + w_2^2)^{3/2}} + \frac{2k_2}{w_2^3},$$

we make the simplifying substitution $w_2 = \sqrt{r_2^2 - w_1^2}$. Then, deriving p_1 we obtain,

$$\dddot{w}_1 = -\frac{k_1}{r_2^3}.$$

Deriving p_2 we obtain:

$$\ddot{r}_2 = \frac{w_1^2 \dot{r}_2^2}{r_2(r_2^2 - w_1^2)} - \frac{2w_1 \dot{w}_1 \dot{r}_2}{r_2^2 - w_1^2} + \frac{\dot{w}_1^2 r_2^3 + 2k_1 w_1 + 2k_2 r_2}{r_2^2(r_2^2 - w_1^2)}.$$

The system w_1, r_2 admits a 3-dim Lie symmetry algebra $sl(2, \mathbb{R})$ generated by:

$$t^2 \partial_t + tw_1 \partial_{w_1} + tr_2 \partial_{r_2}, \quad 2t\partial_t + w_1 \partial_{w_1} + r_2 \partial_{r_2}, \quad \partial_t.$$

If we solve system

$$
\begin{aligned}
\ddot{w}_1 &= -\frac{k_1}{r_2^3}, \\
\ddot{r}_2 &= \frac{w_1^2 \dot{r}_2^2}{r_2(r2^2 - w_1^2)} - \frac{2w_1 \dot{w}_1 \dot{r}_2}{r2^2 - w_1^2} + \frac{\dot{w}_1^2 r_2^3 + 2k_1 w_1 + 2k_2 r_2}{r_2^2(r2^2 - w_1^2)},
\end{aligned}
$$

with respect to k_1, k_2 and derive once, we obtain:

$$
\dddot{w}_1 = -\frac{3\dot{w}_1 \ddot{w}_1}{w_1}, \qquad \dddot{r}_2 = -\frac{3\dot{r}_2 \ddot{r}_2}{r_2},
$$

namely both w_1 and r_2 satisfy the same equation as w_3.

If we solve system

$$\ddot{w}_1 = -\frac{k_1}{r_2^3},$$

$$\ddot{r}_2 = \frac{w_1^2 \dot{r}_2^2}{r_2(r2^2 - w_1^2)} - \frac{2w_1\dot{w}_1\dot{r}_2}{r2^2 - w_1^2} + \frac{\dot{w}_1^2 r_2^3 + 2k_1 w_1 + 2k_2 r_2}{r_2^2(r2^2 - w_1^2)},$$

with respect to k_1, k_2 and derive once, we obtain:

$$\dddot{w}_1 = -\frac{3\dot{w}_1\ddot{w}_1}{w_1}, \qquad \dddot{r}_2 = -\frac{3\dot{r}_2\ddot{r}_2}{r_2},$$

namely both w_1 and r_2 satisfy the same equation as w_3.
Consequently, the transformations $u_1 = w_1^2/2, u_2 = r_2^2/2$ yield:

$$\dddot{u}_1 = 0, \qquad \dddot{u}_2 = 0.$$

If we solve system

$$\ddot{w}_1 = -\frac{k_1}{r_2^3},$$

$$\ddot{r}_2 = \frac{w_1^2 \dot{r}_2^2}{r_2(r2^2 - w_1^2)} - \frac{2w_1 \dot{w}_1 \dot{r}_2}{r2^2 - w_1^2} + \frac{\dot{w}_1^2 r_2^3 + 2k_1 w_1 + 2k_2 r_2}{r_2^2(r2^2 - w_1^2)},$$

with respect to k_1, k_2 and derive once, we obtain:

$$\dddot{w}_1 = -\frac{3\dot{w}_1 \ddot{w}_1}{w_1}, \qquad \dddot{r}_2 = -\frac{3\dot{r}_2 \ddot{r}_2}{r_2},$$

namely both w_1 and r_2 satisfy the same equation as w_3.
Consequently, the transformations $u_1 = w_1^2/2, u_2 = r_2^2/2$ yield:

$$\dddot{u}_1 = 0, \qquad \dddot{u}_2 = 0.$$

Indeed, the Hamiltonian system H_3 hides (three times) the linear
equation $\dddot{u} = 0$.

If we solve system

$$\ddot{w}_1 = -\frac{k_1}{r_2^3},$$

$$\ddot{r}_2 = \frac{w_1^2 \dot{r}_2^2}{r_2(r2^2 - w_1^2)} - \frac{2w_1 \dot{w}_1 \dot{r}_2}{r2^2 - w_1^2} + \frac{\dot{w}_1^2 r_2^3 + 2k_1 w_1 + 2k_2 r_2}{r_2^2(r2^2 - w_1^2)},$$

with respect to k_1, k_2 and derive once, we obtain:

$$\dddot{w}_1 = -\frac{3\dot{w}_1 \ddot{w}_1}{\quad} \qquad \dddot{r}_2 = -\frac{3\dot{r}_2 \ddot{r}_2}{\quad}$$

CONJECTURE:
ARE ALL MAXIMALLY SUPERINTEGRABLE SYSTEMS IN 3-DIM
LINEARIZABLE?

$$\dddot{u}_1 = 0, \qquad \dddot{u}_2 = 0.$$

Indeed, the Hamiltonian system H_3 hides (three times) the linear
equation $\dddot{u} = 0$.

Lagrange vindicated

In the Avertissement to his "Méchanique Analitique" (1788)
Joseph-Louis Lagrange (1736-1813) wrote:

> The methods that I explain in it require neither constructions nor geometrical or mechanical arguments, but only the algebraic operations inherent to a regular and uniform process. Those who love Analysis will, with joy, see mechanics become a new branch of it and will be grateful to me for thus having extended its field. (tr. by J.R. Maddox:)

Lagrange vindicated

In the Avertissement to his "Méchanique Analitique" (1788)
Joseph-Louis Lagrange (1736-1813) wrote:

> The methods that I explain in it require neither constructions nor geometrical or mechanical arguments, but only the algebraic operations inherent to a regular and uniform process. Those who love Analysis will, with joy, see mechanics become a new branch of it and will be grateful to me for thus having extended its field. (tr. by J.R. Maddox:)

It is a joke, isn't it??!!

Jacobi last multiplier

(Jacobi, 1842-45)

$$Af = \sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i} = 0 \qquad (\star)$$

$$\frac{\mathrm{d}x_1}{a_1} = \frac{\mathrm{d}x_2}{a_2} = \ldots = \frac{\mathrm{d}x_n}{a_n}. \quad (\star\star)$$

$$\frac{\partial(f, \omega_1, \omega_2, \ldots, \omega_{n-1})}{\partial(x_1, x_2, \ldots, x_n)} = MAf$$

$\omega_i, (i = 1, \ldots, n-1)$ solutions of $(\star)$ ie first integrals of $(\star\star)$

$$\sum_{i=1}^{n} \frac{\partial(Ma_i)}{\partial x_i} = 0 \Leftrightarrow \frac{\mathrm{d}\log(M)}{\mathrm{d}t} = -\sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i}$$

Jacobi last multiplier

(Jacobi, 1842-45)

$$Af = \sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i} = 0 \qquad (\star)$$

$$\frac{\mathrm{d}x_1}{a_1} = \frac{\mathrm{d}x_2}{a_2} = \ldots = \frac{\mathrm{d}x_n}{a_n}. \quad (\star\star)$$

$$\frac{\partial(f, \omega_1, \omega_2, \ldots, \omega_{n-1})}{\partial(x_1, x_2, \ldots, x_n)} = MAf$$

$\omega_i, (i = 1, \ldots, n-1)$ solutions of $(\star)$ ie first integrals of $(\star\star)$

$$\sum_{i=1}^{n} \frac{\partial(Ma_i)}{\partial x_i} = 0 \Leftrightarrow \frac{\mathrm{d}\log(M)}{\mathrm{d}t} = -\sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i}$$

IMPORTANT PROPERTY:

$$\frac{M_1}{M_2} = \text{First Integral}$$

Enter Lie

[Lie, 1874]

$$\frac{\mathrm{d}x_1}{a_1} = \frac{\mathrm{d}x_2}{a_2} = \ldots = \frac{\mathrm{d}x_n}{a_n}. \quad (\star\star)$$

If there exist $n-1$ symmetries of $(\star\star)$, say

$$\Gamma_i = \xi_{ij}\partial_{x_j}, \quad i = 1, n-1$$

then JLM is given by $M = \Delta^{-1}$, provided that $\Delta \neq 0$, where

$$\Delta = \det \begin{bmatrix} a_1 & \cdots & a_n \\ \xi_{1,1} & & \xi_{1,n} \\ \vdots & & \vdots \\ \xi_{n-1,1} & \cdots & \xi_{n-1,n} \end{bmatrix}$$

Corollary: if $\exists$ M=const, then Δ is a first integral.

How many Lagrangians does one know?

There is a link between a Jacobi Last Multiplier M and a Lagrangian L [Jacobi, 1842-45], [also in Whittaker, 1904].

Jacobi's Lectures on Dynamics (1884) are available in English: tr. by K. Balagangadharan, ed. by Biswarup Banerjee, Hindustan Book Agency (2009), available through AMS

For a second-order ODE the link is:

$$\frac{\partial^2 L}{\partial \dot{q}^2} = M. \tag{1}$$

Consequently a knowledge of the multipliers of a system enables one to construct a number of Lagrangians of that system.

How many??

How many Lagrangians does one know?

There is a link between a Jacobi Last Multiplier M and a Lagrangian L [Jacobi, 1842-45], [also in Whittaker, 1904].

Jacobi's Lectures on Dynamics (1884) are available in English: tr. by K. Balagangadharan, ed. by Biswarup Banerjee, Hindustan Book Agency (2009), available through AMS

For a second-order ODE the link is:

$$\frac{\partial^2 L}{\partial \dot{q}^2} = M. \tag{1}$$

Consequently a knowledge of the multipliers of a system enables one to construct a number of Lagrangians of that system.

How many??

N.B.: For a single ODE of order $2n$ the link is $M^{1/n} = \dfrac{\partial^2 L}{\partial \left(q^{(n)}\right)^2}$

(Jacobi, *J. Reine Angew. Math.* **29** (1845) p.364)

A very simple example

Let us consider the one-dimensional free particle $\ddot{x} = 0$, i.e.:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = 0$$

Lie symmetry algebra $sl(3, \mathbb{R})$:

$$X_1 = xt\partial_t + x^2\partial_x, \quad X_2 = x\partial_t, \quad X_3 = t^2\partial_t + xt\partial_x, \quad X_4 = x\partial_x,$$

$$X_5 = t\partial_t, \quad X_6 = \partial_t, \quad X_7 = t\partial_x, \quad X_8 = \partial_x.$$

A very simple example

Let us consider the one-dimensional free particle $\ddot{x} = 0$, i.e.:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = 0$$

Lie symmetry algebra $sl(3, \mathbb{R})$:

$$X_1 = xt\partial_t + x^2\partial_x, \quad X_2 = x\partial_t, \quad X_3 = t^2\partial_t + xt\partial_x, \quad X_4 = x\partial_x,$$

$$X_5 = t\partial_t, \quad X_6 = \partial_t, \quad X_7 = t\partial_x, \quad X_8 = \partial_x.$$

$JLM_{ij} = 1/\Delta_{ij}$, X_i and X_j

A very simple example

Let us consider the one-dimensional free particle $\ddot{x} = 0$, i.e.:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = 0$$

Lie symmetry algebra $sl(3, \mathbb{R})$:

$$X_1 = xt\partial_t + x^2\partial_x, \quad X_2 = x\partial_t, \quad X_3 = t^2\partial_t + xt\partial_x, \quad X_4 = x\partial_x,$$

$$X_5 = t\partial_t, \quad X_6 = \partial_t, \quad X_7 = t\partial_x, \quad X_8 = \partial_x.$$

$JLM_{ij} = 1/\Delta_{ij}$, X_i and X_j
For example $JLM_{48} = -1/\dot{x}$ by means of X_4 and X_8 such that:

$$\Delta_{48} = \det \begin{bmatrix} 1 & x_2 & 0 \\ 0 & x_1 & x_2 \\ 0 & 1 & 0 \end{bmatrix} = -x_2 \equiv -\dot{x}.$$

Ten Lagrangians

Ten different JLM and consequently as many Lagrangians:

$$M_{13} = -\frac{1}{(t\dot{x} - x)^3} \quad \Rightarrow \quad L_{1,3} = -\frac{1}{2t^2(t\dot{x} - x)} + \frac{\mathrm{d}g}{\mathrm{d}t}(t, x)$$

$$M_{15} = -\frac{1}{\dot{x}(t\dot{x} - x)^2} \quad \Rightarrow \quad L_{1,5} = \frac{\dot{x}}{x^2}\left(\log(t\dot{x} - x) - \log(\dot{x})\right)$$

$$M_{16} = \frac{1}{\dot{x}^2(t\dot{x} - x)} \quad \Rightarrow \quad L_{1,6} = \left(\frac{t\dot{x}}{x^2} - \frac{1}{x}\right)\left(\log(\dot{x}) - \log(t\dot{x} - x)\right)$$

$$M_{17} = -\frac{1}{(t\dot{x} - x)^2} \quad \Rightarrow \quad L_{1,7} = -\frac{1}{t^2}\log(t\dot{x} - x)$$

$$M_{18} = \frac{1}{\dot{x}(t\dot{x} - x)} \quad \Rightarrow \quad L_{1,8} = -\frac{\dot{x}}{x}\log(\dot{x}) - \left(\frac{1}{t} - \frac{\dot{x}}{x}\right)\log(t\dot{x} - x)$$
$$+ \frac{1}{t}(1 + \log(x))$$

$$M_{62} = \frac{1}{\dot{x}^3} \quad \Rightarrow \quad L_{6,2} = \frac{1}{2\dot{x}}$$

$$M_{28} = \frac{1}{\dot{x}^2} \quad \Rightarrow \quad L_{2,8} = -\log(\dot{x})$$

$$M_{38} = \frac{1}{t\dot{x} - x} \quad \Rightarrow \quad L_{3,8} = \left(\frac{\dot{x}}{t} - \frac{x}{t^2} \right) \left(\log(t\dot{x} - x) - 1 \right)$$

$$M_{48} = -\frac{1}{\dot{x}} \quad \Rightarrow \quad L_{4,8} = \dot{x}(1 - \log(\dot{x}))$$

$$M_{87} = 1 \quad \Rightarrow$$

$$M_{62} = \frac{1}{\dot{x}^3} \quad \Rightarrow \quad L_{6,2} = \frac{1}{2\dot{x}}$$

$$M_{28} = \frac{1}{\dot{x}^2} \quad \Rightarrow \quad L_{2,8} = -\log(\dot{x})$$

$$M_{38} = \frac{1}{t\dot{x} - x} \quad \Rightarrow \quad L_{3,8} = \left(\frac{\dot{x}}{t} - \frac{x}{t^2} \right) (\log(t\dot{x} - x) - 1)$$

$$M_{48} = -\frac{1}{\dot{x}} \quad \Rightarrow \quad L_{4,8} = \dot{x}(1 - \log(\dot{x}))$$

$$M_{87} = 1 \quad \Rightarrow \quad L_{8,7} = \frac{1}{2}\dot{x}^2$$

FINALLY, THE TRUE LAGRANGIAN

How do we (physically) eliminate 9 out of 10??

How do we (physically) eliminate 9 out of 10??

- They differ by the number of Noether symmetries that they admit.

How do we (physically) eliminate 9 out of 10??

- They differ by the number of Noether symmetries that they admit.
- The physical Lagrangian admits the maximum number of Noether symmetries, i.e. FIVE.

How do we (physically) eliminate 9 out of 10??

- They differ by the num   ey
  admit.
- The physical Lagrangia
  Noether symmetries, i.e

EMMY NOETHER

How do we (physically) eliminate 9 out of 10??

ERWIN SCHRÖDINGER

EMMY NOETHER

Which Quantization?

Different methods exist to describe the dynamics of quantum systems, among them:

Which Quantization?

Different methods exist to describe the dynamics of quantum systems, among them:

> **(i)** the time-dependent Schrödinger equation with the corresponding continuity equation for the probability density;

Which Quantization?

Different methods exist to describe the dynamics of quantum systems, among them:

(i) the time-dependent Schrödinger equation with the corresponding continuity equation for the probability density;

(ii) the description using a time propagator, also called Feynman kernel or space representation of the Green function;

Which Quantization?

Different methods exist to describe the dynamics of quantum systems, among them:

    (i) the time-dependent Schrödinger equation with the corresponding continuity equation for the probability density;

    (ii) the description using a time propagator, also called Feynman kernel or space representation of the Green function;

    (iii) the time-dependent Wigner function.

Which Quantization?

Different methods exist to describe the dynamics of quantum systems, among them:

   (i) the time-dependent Schrödinger equation with the corresponding continuity equation for the probability density;

  (ii) the description using a time propagator, also called Feynman kernel or space representation of the Green function;

 (iii) the time-dependent Wigner function.

All three methods were shown to be linked by the Ermakov invariant in *D. Schuch & M. Moshinsky, Phys.Rev.A, 2006*.

Which Quantization?

Different methods exist to describe the dynamics of quantum systems, among them:

(i) the time-dependent Schrödinger equation with the corresponding continuity equation for the probability density;

(ii) the description using a time propagator, also called Feynman kernel or space representation of the Green function;

(iii) the time-dependent Wigner function.

All three methods were shown to be linked by the Ermakov invariant in *D. Schuch & M. Moshinsky, Phys.Rev.A, 2006*. Therefore we pursue the quantization of classical problems by searching for a time-dependent Schrödinger equation.

How to obtain the Schrödinger equation from Noether symmetries I

$$\ddot{q} = 0$$

$L = \frac{1}{2}\dot{q}^2$ admits five Noether symmetries:

$X_1 = \partial_t,\ X_2 = \partial_q,\ X_3 = t\partial_q,\ X_4 = 2t\partial_t + q\partial_q,\ X_5 = t^2\partial_t + tq\partial_q.$

How to obtain the Schrödinger equation from Noether symmetries I

$$\ddot{q} = 0$$

$L = \frac{1}{2}\dot{q}^2$ admits five Noether symmetries:

$$X_1 = \partial_t, \ X_2 = \partial_q, \ X_3 = t\partial_q, \ X_4 = 2t\partial_t + q\partial_q, \ X_5 = t^2\partial_t + tq\partial_q.$$

The Schrödinger equation

$$2i\psi_t + \psi_{qq} = 0$$

admits $5 + 1 + \infty$ Lie symmetries:

$$Y_1 = X_1, \ Y_2 = X_2, \ Y_3 = X_3 + iq\psi\partial_\psi, \ Y_4 = X_4,$$
$$Y_5 = X_5 + \tfrac{1}{2}(iq^2 - t)\psi\partial_\psi.$$

plus $\psi\partial_\psi$ and $\alpha(t, q)\partial_\psi$ s.t. $2i\alpha_t + \alpha_{qq} = 0$

Marcos Moshinsky

He was used to say [K.B. Wolf, J. Phys.: Conf. Ser. 237 (2010)]:

Marcos Moshinsky

He was used to say [K.B. Wolf, J. Phys.: Conf. Ser. 237 (2010)]:

Two types of problems exist in quantum mechanics, those that you cannot solve and the harmonic oscillator. The trick is to push a problem from one category to the other.

How to obtain the Schrödinger equation from Noether symmetries II

$$\ddot{x} = -\omega^2 x$$

How to obtain the Schrödinger equation from Noether symmetries II

$$\ddot{x} = -\omega^2 x$$

$L = \frac{1}{2}(\dot{x}^2 - \omega^2 x^2)$ admits 5 Noether symmetries:

$$
\begin{aligned}
X_1 &= \partial_t, \ X_2 = \cos(2\omega t)\partial_t - \omega x \sin(2\omega t)\partial_x, \\
X_3 &= \sin(2\omega t)\partial_t + \omega x \cos(2\omega t)\partial_x, \ X_4 = \cos(\omega t)\partial_x, \\
X_5 &= \sin(\omega t)\partial_x.
\end{aligned}
$$

How to obtain the Schrödinger equation from Noether symmetries II

$$\ddot{x} = -\omega^2 x$$

$L = \frac{1}{2}(\dot{x}^2 - \omega^2 x^2)$ admits 5 Noether symmetries:

$$
\begin{aligned}
X_1 &= \partial_t,\ X_2 = \cos(2\omega t)\partial_t - \omega x \sin(2\omega t)\partial_x, \\
X_3 &= \sin(2\omega t)\partial_t + \omega x \cos(2\omega t)\partial_x,\ X_4 = \cos(\omega t)\partial_x, \\
X_5 &= \sin(\omega t)\partial_x.
\end{aligned}
$$

The Schrödinger equation (with $\hbar = 1$)

$$2i\psi_t + \psi_{xx} - \omega^2 x^2 \psi = 0$$

How to obtain the Schrödinger equation from Noether symmetries II

$$\ddot{x} = -\omega^2 x$$

$L = \frac{1}{2}(\dot{x}^2 - \omega^2 x^2)$ admits 5 Noether symmetries:

$$
\begin{aligned}
X_1 &= \partial_t, \ X_2 = \cos(2\omega t)\partial_t - \omega x \sin(2\omega t)\partial_x, \\
X_3 &= \sin(2\omega t)\partial_t + \omega x \cos(2\omega t)\partial_x, \ X_4 = \cos(\omega t)\partial_x, \\
X_5 &= \sin(\omega t)\partial_x.
\end{aligned}
$$

The Schrödinger equation (with $\hbar = 1$)

$$2i\psi_t + \psi_{xx} - \omega^2 x^2 \psi = 0$$

admits the $5 + 1 + \infty$ dimensional Lie symmetry algebra:

$$
\begin{aligned}
Y_1 &= X_1, \ Y_2 = X_2 + \omega(\sin(2\omega t) - 2i\omega x^2 \cos(2\omega t))\psi\partial_\psi, \\
Y_3 &= X_3 - \omega(\cos(2\omega t) - 2i\omega x^2 \sin(2\omega t))\psi\partial_\psi, \\
Y_4 &= X_4 - 2i\omega x \sin(\omega t)\psi\partial_\psi, \ Y_5 = X_5 + 2i\omega x \cos(\omega t)\psi\partial_\psi.
\end{aligned}
$$

plus $\psi\partial_\psi$ and $\alpha(t,x)\partial_\psi$ s.t. $2i\alpha_t + \alpha_{xx} - \omega^2 x^2 \alpha = 0$

Quantizing with Noether: Act 1

MCN, Theor.Math.Phys., 2011

Quantizing with Noether: Act 1

- Find the Lie symmetries of the Lagrange equation

$$\Upsilon = W_0(t,x)\partial_t + W_1(t,x)\partial_x$$

Quantizing with Noether: Act 1

- Find the Lie symmetries of the Lagrange equation

$$\Upsilon = W_0(t,x)\partial_t + W_1(t,x)\partial_x$$

- Find the Noether symmetries

$$\Gamma = V_0(t,x)\partial_t + V_1(t,x)\partial_x, \qquad \Gamma \in \Upsilon$$

Quantizing with Noether: Act 1

- Find the Lie symmetries of the Lagrange equation

$$\Upsilon = W_0(t,x)\partial_t + W_1(t,x)\partial_x$$

- Find the Noether symmetries

$$\Gamma = V_0(t,x)\partial_t + V_1(t,x)\partial_x, \qquad \Gamma \in \Upsilon$$

- Construct the Schrödinger equation admitting those Lie symmetries

$$2i\psi_t + f_1(x)\psi_{xx} + f_2(x)\psi_x + f_3(x)\psi = 0$$

$$\Omega = V_0(t,x)\partial_t + V_1(t,x)\partial_x + G(t,x,\psi)\partial_\psi$$

Quantizing with Noether: Act 1

- Find the Lie symmetries of the Lagrange equation

$$\Upsilon = W_0(t,x)\partial_t + W_1(t,x)\partial_x$$

- Find the Noether symmetries

$$\Gamma = V_0(t,x)\partial_t + V_1(t,x)\partial_x, \qquad \Gamma \in \Upsilon$$

- Construct the Schrödinger equation admitting those Lie symmetries

$$2i\psi_t + f_1(x)\psi_{xx} + f_2(x)\psi_x + f_3(x)\psi = 0$$

$$\Omega = V_0(t,x)\partial_t + V_1(t,x)\partial_x + G(t,x,\psi)\partial_\psi$$

- N.B. $\psi\partial_\psi$ and $\Psi(t,x)\partial_\psi$

Quantizing with Noether: Act 1

- Find the Lie symmetries of the Lagrange equation

$$\Upsilon = W_0(t,x)\partial_t + W_1(t,x)\partial_x$$

- Find the Noether symmetries

$$\Gamma = V_0(t,x)\partial_t + V_1(t,x)\partial_x, \qquad \Gamma \in \Upsilon$$

- Construct the Schrödinger equation admitting those Lie symmetries

$$2i\psi_t + f_1(x)\psi_{xx} + f_2(x)\psi_x + f_3(x)\psi = 0$$

$$\Omega = V_0(t,x)\partial_t + V_1(t,x)\partial_x + G(t,x,\psi)\partial_\psi$$

QUANTIZE PRESERVING THE SYMMETRIES!

An example from population dynamics

Equation

$$\ddot{u} = \frac{5\dot{u}^2}{4u} - \frac{2c^2}{K}u^2 - c^2 u$$

admits a 3-dim Lie (**complete**) symmetry group: ★★★

$$\Gamma_1 = \partial_t, \qquad \Gamma_2 = e^{-ct}\left(\partial_t + 2cu\partial_u\right) \qquad \Gamma_3 = e^{ct}\left(\partial_t - 2cu\partial_u\right).$$

An example from population dynamics

MCN & G. Sanchini, Symmetry, 2016 Equation

$$\ddot{u} = \frac{5\dot{u}^2}{4u} - \frac{2c^2}{K}u^2 - c^2 u$$

admits a 3-dim Lie (**complete**) symmetry group: ★★★

$$\Gamma_1 = \partial_t, \qquad \Gamma_2 = e^{-ct}\left(\partial_t + 2cu\partial_u\right) \qquad \Gamma_3 = e^{ct}\left(\partial_t - 2cu\partial_u\right).$$

The Lagrangian with those 3 Noether symmetries is

$$L = \sqrt{u}\left(\frac{\dot{u}^2}{4cu^3} + \frac{c}{u} - \frac{2c}{K}\right)$$

An example from population dynamics

MCN & G. Sanchini, Symmetry, 2016 Equation

$$\ddot{u} = \frac{5\dot{u}^2}{4u} - \frac{2c^2}{K}u^2 - c^2 u$$

admits a 3-dim Lie (**complete**) symmetry group: ★★★

$$\Gamma_1 = \partial_t, \qquad \Gamma_2 = e^{-ct}\left(\partial_t + 2cu\partial_u\right) \qquad \Gamma_3 = e^{ct}\left(\partial_t - 2cu\partial_u\right).$$

The Lagrangian with those 3 Noether symmetries is

$$L = \sqrt{u}\left(\frac{\dot{u}^2}{4cu^3} + \frac{c}{u} - \frac{2c}{K}\right)$$

and the Schrödinger equation is

$$2i\psi_t + u^2\sqrt{u}\psi_{uu} - \sqrt{u}\left(\eta + 4\frac{c^2}{u}\right)\psi = 0,$$

An example from population dynamics

MCN & G. Sanchini, Symmetry, 2016 Equation

$$\ddot{u} = \frac{5\dot{u}^2}{4u} - \frac{2c^2}{K}u^2 - c^2 u$$

admits a 3-dim Lie (**complete**) symmetry group: ★★★

$$\Gamma_1 = \partial_t, \qquad \Gamma_2 = e^{-ct}\left(\partial_t + 2cu\partial_u\right) \qquad \Gamma_3 = e^{ct}\left(\partial_t - 2cu\partial_u\right).$$

The Lagrangian with those 3 Noether symmetries is

$$L = \sqrt{u}\left(\frac{\dot{u}^2}{4cu^3} + \frac{c}{u} - \frac{2c}{K}\right)$$

and the Schrödinger equation is

$$2\mathrm{i}\psi_t + u^2\sqrt{u}\psi_{uu} - \sqrt{u}\left(\eta + 4\frac{c^2}{u}\right)\psi = 0,$$

with : $\quad \Lambda_1 = \Gamma_1, \qquad \Lambda_2 = \Gamma_2 + ce^{-ct}\left(\frac{3}{2} + 4\mathrm{i}\frac{c}{\sqrt{u}}\right)\psi\partial_\psi,$

$$\Lambda_3 = \Gamma_3 + ce^{ct}\left(-\frac{3}{2} + 4\mathrm{i}\frac{c}{\sqrt{u}}\right)\psi\partial_\psi.$$

Charged particle in a uniform magnetic field

Its classical Lagrangian is

$$L = \frac{1}{2}\left((\dot{x}^2 + \dot{y}^2) + \omega(y\dot{x} - x\dot{y})\right) \tag{2}$$

and consequently the Lagrangian equations are

$$\ddot{x} = -\omega\dot{y}, \quad \ddot{y} = \omega\dot{x}.$$

Charged particle in a uniform magnetic field

Its classical Lagrangian is

$$L = \frac{1}{2}\left((\dot{x}^2 + \dot{y}^2) + \omega(y\dot{x} - x\dot{y})\right) \tag{2}$$

and consequently the Lagrangian equations are

$$\ddot{x} = -\omega\dot{y}, \quad \ddot{y} = \omega\dot{x}.$$

The Lagrangian (2) admits 8 Noether symmetries generated by

$$
\begin{aligned}
X_1 &= \cos(\omega t)\partial_t - \frac{\omega}{2}\left(\sin(\omega t)x + \cos(\omega t)y\right)\partial_x \\
&\quad + \frac{\omega}{2}\left(\cos(\omega t)x - \sin(\omega t)y\right)\partial_y, \\
X_2 &= -\sin(\omega t)\partial_t - \frac{1}{2}\left(\cos(\omega t)\omega x - \sin(\omega t)\omega y\right)\partial_x \\
&\quad - \frac{1}{2}(\sin(\omega t)\omega x + \cos(\omega t)\omega y)\partial_y, \\
X_3 &= \partial_t, \quad X_4 = -y\partial_x + x\partial_y, \quad X_5 = -\sin(\omega t)\partial_x + \cos(\omega t)\partial_y, \\
X_6 &= -\cos(\omega t)\partial_x - \sin(\omega t)\partial_y, \quad X_7 = \partial_y, \quad X_8 = \partial_x.
\end{aligned}
$$

The Schrödinger equation was determined by Sir Charles Galton Darwin, Proc. R. Soc. Lond. A, 1927 to be

$$2i\psi_t + \psi_{xx} + \psi_{yy} - i\omega(y\psi_x - x\psi_y) - \frac{\omega^2}{4}(x^2 + y^2)\psi = 0. \quad (3)$$

The Schrödinger equation was determined by Sir Charles Galton Darwin, Proc. R. Soc. Lond. A, 1927 to be

$$2i\psi_t + \psi_{xx} + \psi_{yy} - i\omega(y\psi_x - x\psi_y) - \frac{\omega^2}{4}(x^2 + y^2)\psi = 0. \quad (3)$$

It admits the $8 + 1 + \infty$ dim Lie symmetry algebra, i.e.:

$$
\begin{aligned}
Y_1 &= X_1 + \frac{1}{4}\left(2\sin(\omega t)\omega - i\cos(\omega t)\omega^2(x^2 + y^2)\right)\partial_\psi, \\
Y_2 &= X_2 + \frac{1}{4}\left(2\cos(\omega t)\omega + i\sin(\omega t)\omega^2(x^2 + y^2)\right)\partial_\psi, \\
Y_3 &= X_3, \quad Y_4 = X_4, \quad Y_5 = X_5 - \frac{1}{2}\omega\left(x\cos(\omega t) + y\sin(\omega t)\right)\partial_\psi, \\
Y_6 &= X_6 + \frac{1}{2}\omega\left(x\sin(\omega t) - y\cos(\omega t)\right)\partial_\psi, \quad Y_7 = X_7 + \frac{i}{2}\omega x\partial_\psi, \\
Y_8 &= X_8 - \frac{i}{2}\omega y\partial_\psi, \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (4)
\end{aligned}
$$

plus $\psi\partial_\psi$ and $\alpha(t,x,y)\partial_\psi$ s.t. α satisfies (3).

Quantizing with Noether: Act 2

Quantizing with Noether: Act 2

- Find the Lie symmetries of the Lagrange equations

$$\Upsilon = W(t, \underline{x})\partial_t + \sum_{k=1}^{N} W_k(t, \underline{x})\partial_{x_k}$$

Quantizing with Noether: Act 2

- Find the Lie symmetries of the Lagrange equations

$$\Upsilon = W(t, \underline{x})\partial_t + \sum_{k=1}^{N} W_k(t, \underline{x})\partial_{x_k}$$

- Find the Noether symmetries

$$\Gamma = V(t, \underline{x})\partial_t + \sum_{k=1}^{N} V_k(t, \underline{x})\partial_{x_k}, \quad \Gamma \subset \Upsilon$$

Quantizing with Noether: Act 2

- Find the Lie symmetries of the Lagrange equations

$$\Upsilon = W(t,\underline{x})\partial_t + \sum_{k=1}^{N} W_k(t,\underline{x})\partial_{x_k}$$

- Find the Noether symmetries

$$\Gamma = V(t,\underline{x})\partial_t + \sum_{k=1}^{N} V_k(t,\underline{x})\partial_{x_k}, \quad \Gamma \subset \Upsilon$$

- Construct the Schrödinger equation admitting those Lie symmetries

$$2i\psi_t + \sum_{k,j=1}^{N} f_{kj}(\underline{x})\psi_{x_j x_k} + \sum_{k=1}^{N} h_k(\underline{x})\psi_{x_k} + f_3(\underline{x})\psi = 0$$

$$\Omega = V(t,\underline{x})\partial_t + \sum_{k=1}^{N} V_k(t,\underline{x})\partial_{x_k} + G(t,\underline{x},\psi)\partial_\psi$$

Quantizing with Noether: Act 2

- Find the Lie symmetries of the Lagrange equations

$$\Upsilon = W(t, \underline{x})\partial_t + \sum_{k=1}^{N} W_k(t, \underline{x})\partial_{x_k}$$

- Find the Noether symmetries

$$\Gamma = V(t, \underline{x})\partial_t + \sum_{k=1}^{N} V_k(t, \underline{x})\partial_{x_k}, \quad \Gamma \subset \Upsilon$$

- Construct the Schrödinger equation admitting those Lie symmetries

$$2i\psi_t + \sum_{k,j=1}^{N} f_{kj}(\underline{x})\psi_{x_j x_k} + \sum_{k=1}^{N} h_k(\underline{x})\psi_{x_k} + f_3(\underline{x})\psi = 0$$

$$\Omega = V(t, \underline{x})\partial_t + \sum_{k=1}^{N} V_k(t, \underline{x})\partial_{x_k} + G(t, \underline{x}, \psi)\partial_\psi$$

Quantizing with Noether: Act 2

- Find the Lie symmetries of the Lagrange equations

$$\Upsilon = W(t, \underline{x})\partial_t + \sum_{k=1}^{N} W_k(t, \underline{x})\partial_{x_k}$$

- Find the Noether symmetries

$$\Gamma = V(t, \underline{x})\partial_t + \sum_{k=1}^{N} V_k(t, \underline{x})\partial_{x_k}, \quad \Gamma \subset \Upsilon$$

- Construct the Schrödinger equation admitting those Lie symmetries

$$2i\psi_t + \sum_{k,j=1}^{N} f_{kj}(\underline{x})\psi_{x_j x_k} + \sum_{k=1}^{N} h_k(\underline{x})\psi_{x_k} + f_3(\underline{x})\psi = 0$$

QUANTIZE PRESERVING THE SYMMETRIES!

N charged particles in a uniform magnetic field

MCN, Theor.Math.Phys., 2016 In complex variables the Newtonian eqs are:

$$\ddot{r}_k = i\omega \dot{r}_k, \quad (k = 1, \ldots, N).$$

N charged particles in a uniform magnetic field

MCN, Theor.Math.Phys., 2016 In complex variables the Newtonian eqs are:

$$\ddot{r}_k = i\omega \dot{r}_k, \quad (k = 1, \ldots, N).$$

Among many Lagrangians, let us consider the following:

$$L_0 = \frac{\exp(-i\omega t)}{2} \sum_{k=1}^{N} \dot{r}_k^2,$$

that admits $(N^2 + 3N + 6)/2$ Noether symmetries.

N charged particles in a uniform magnetic field

MCN, Theor.Math.Phys., 2016 In complex variables the Newtonian eqs are:

$$\ddot{r}_k = i\omega \dot{r}_k, \quad (k = 1, \ldots, N).$$

Among many Lagrangians, let us consider the following:

$$L_0 = \frac{\exp(-i\omega t)}{2} \sum_{k=1}^{N} \dot{r}_k^2,$$

that admits $(N^2 + 3N + 6)/2$ Noether symmetries.
Yet there is another time-independent Lagrangian:

$$L = \sum_{k=1}^{N} \left(\frac{1}{2} \dot{r}_k \dot{\widetilde{r}}_k + \frac{i\omega}{4}(\widetilde{r}_k \dot{r}_k - r_k \dot{\widetilde{r}}_k) \right)$$

that also admits $(N^2 + 3N + 6)/2$ Noether symmetries.

Replacing

$$r_k = x_k + iy_k, \quad \widetilde{r}_k = x_k - iy_k$$

into the Lagrangian L one gets:

$$L = \sum_{k=1}^{N} \left(\frac{1}{2}(\dot{x}_k^2 + \dot{y}_k^2) + \frac{\omega}{2}(y_k\dot{x}_k - x_k\dot{y}_k) \right),$$

and the Lagrangian equations are:

$$\ddot{x}_k = -\omega\dot{y}_k, \quad \ddot{y}_k = \omega\dot{x}_k.$$

Replacing

$$r_k = x_k + iy_k, \quad \widetilde{r}_k = x_k - iy_k$$

into the Lagrangian L one gets:

$$L = \sum_{k=1}^{N} \left(\frac{1}{2}(\dot{x}_k^2 + \dot{y}_k^2) + \frac{\omega}{2}(y_k\dot{x}_k - x_k\dot{y}_k) \right),$$

and the Lagrangian equations are:

$$\ddot{x}_k = -\omega\dot{y}_k, \quad \ddot{y}_k = \omega\dot{x}_k.$$

Then the Schrödinger equation that can be obtained by preserving the symmetries is:

$$2i\psi_t + \sum_{k=1}^{N} \left(\psi_{x_k x_k} + \psi_{y_k y_k} + i\omega(y_k\psi_{x_k} - x_k\psi_{y_k}) - \frac{\omega^2}{4}(x_k^2 + y_k^2)\psi \right) = 0.$$

Replacing

$$r_k = x_k + iy_k, \quad \widetilde{r}_k = x_k - iy_k$$

into the Lagrangian L one gets:

$$L = \sum_{k=1}^{N} \left(\frac{1}{2}(\dot{x}_k^2 + \dot{y}_k^2) + \frac{\omega}{2}(y_k \dot{x}_k - x_k \dot{y}_k) \right),$$

and the Lagrangian equations are:

$$\ddot{x}_k = -\omega \dot{y}_k, \quad \ddot{y}_k = \omega \dot{x}_k.$$

Then the Schrödinger equation that can be obtained by preserving the symmetries is:

$$2i\psi_t + \sum_{k=1}^{N} \left(\psi_{x_k x_k} + \psi_{y_k y_k} + i\omega(y_k \psi_{x_k} - x_k \psi_{y_k}) - \frac{\omega^2}{4}(x_k^2 + y_k^2)\psi \right) = 0.$$

Indeed, this eq. admits a $\frac{N^2 + 3N + 6}{2} + 1 + \infty$ dimensional Lie symmetry algebra.

A simplification: the role of $sl(N+2, \mathbb{R})$

In *G.Gubbiotti & MCN, J. Nonlinear Math.Phys., 2014*, if the N Lagrangian equations admit a $(N^2 + 4N + 3)$-dim Lie symmetry algebra, $sl(N+2, \mathbb{R})$, namely they are linearizable by a point transformation, then we simplified the algorithm as follows:

A simplification: the role of $sl(N+2, \mathbb{R})$

In *G.Gubbiotti & MCN, J. Nonlinear Math.Phys., 2014*, if the N Lagrangian equations admit a $(N^2 + 4N + 3)$-dim Lie symmetry algebra, $sl(N+2, \mathbb{R})$, namely they are linearizable by a point transformation, then we simplified the algorithm as follows:

- Find the linearizing transformation.

A simplification: the role of $sl(N+2, \mathbb{R})$

In *G.Gubbiotti & MCN, J. Nonlinear Math.Phys., 2014*, if the N Lagrangian equations admit a $(N^2 + 4N + 3)$-dim Lie symmetry algebra, $sl(N+2, \mathbb{R})$, namely they are linearizable by a point transformation, then we simplified the algorithm as follows:

- Find the linearizing transformation.
- Derive the Lagrangian that admits the maximum number of Noether symmetries by applying the linearizing transformation to the natural Lagrangian of the corresponding linear system.

A simplification: the role of $sl(N+2, \mathbb{R})$

In *G.Gubbiotti & MCN, J. Nonlinear Math.Phys., 2014*, if the N Lagrangian equations admit a $(N^2 + 4N + 3)$-dim Lie symmetry algebra, $sl(N+2, \mathbb{R})$, namely they are linearizable by a point transformation, then we simplified the algorithm as follows:

- Find the linearizing transformation.
- Derive the Lagrangian that admits the maximum number of Noether symmetries by applying the linearizing transformation to the natural Lagrangian of the corresponding linear system.
- Derive the Schrödinger equation by applying the linearizing transformation to the Schrödinger equation of the corresponding linear system.

A simplification: the role of $sl(N + 2, \mathbb{R})$

In *G.Gubbiotti & MCN, J. Nonlinear Math.Phys., 2014*, if the N Lagrangian equations admit a $(N^2 + 4N + 3)$-dim Lie symmetry algebra, $sl(N + 2, \mathbb{R})$, namely they are linearizable by a point transformation, then we simplified the algorithm as follows:

- Find the linearizing transformation.
- Derive the Lagrangian that admits the maximum number of Noether symmetries by applying the linearizing transformation to the natural Lagrangian of the corresponding linear system.
- Derive the Schrödinger equation by applying the linearizing transformation to the Schrödinger equation of the corresponding linear system.

See also *G. Gubbiotti & MCN, J.Math.Anal.Appl., 2015*.

Harmonic oscillator on a double cone

N.B.: $m = 1$, opening angle $2\alpha, \alpha \in \left(0, \frac{\pi}{2}\right), k = \sin(\alpha)$

The Lagrangian $L_{\text{ho}} = \frac{1}{2}\left(\dot{r}^2 + k^2 r^2 \dot{\phi}^2\right) - \frac{1}{2}\omega^2 r^2$ admits 8 Noether symmetries, and the corresponding Lagrangian equations

$$\ddot{r} = k^2 r\dot{\phi}^2 - \omega^2 r, \quad \ddot{\phi} = -2\frac{\dot{r}\dot{\phi}}{r} \qquad (\spadesuit)$$

admit a 15-dim Lie symmetry algebra.

Harmonic oscillator on a double cone

N.B.: $m = 1$, opening angle $2\alpha, \alpha \in \left(0, \frac{\pi}{2}\right), k = \sin(\alpha)$

The Lagrangian $L_{\mathsf{ho}} = \frac{1}{2}\left(\dot{r}^2 + k^2 r^2 \dot{\phi}^2\right) - \frac{1}{2}\omega^2 r^2$ admits 8 Noether symmetries, and the corresponding Lagrangian equations

$$\ddot{r} = k^2 r \dot{\phi}^2 - \omega^2 r, \quad \ddot{\phi} = -2\frac{\dot{r}\dot{\phi}}{r} \qquad (\spadesuit)$$

admit a 15-dim Lie symmetry algebra. Then, the transformation

$$u = r\cos(k\phi), \quad v = r\sin(k\phi) \qquad (\diamondsuit)$$

yields:

$$\ddot{u} + \omega^2 u = 0, \quad \ddot{v} + \omega^2 v = 0,$$

a two-dim harmonic oscillator in (u, v).

Harmonic oscillator on a double cone

G.Gubbiotti & MCN, J. Nonlinear Math.Phys., 2017

N.B.: $m = 1$, opening angle $2\alpha, \alpha \in \left(0, \frac{\pi}{2}\right), k = \sin(\alpha)$

The Lagrangian $L_{\text{ho}} = \frac{1}{2}\left(\dot{r}^2 + k^2 r^2 \dot{\phi}^2\right) - \frac{1}{2}\omega^2 r^2$ admits 8 Noether symmetries, and the corresponding Lagrangian equations

$$\ddot{r} = k^2 r \dot{\phi}^2 - \omega^2 r, \quad \ddot{\phi} = -2\frac{\dot{r}\dot{\phi}}{r} \qquad (\spadesuit)$$

admit a 15-dim Lie symmetry algebra. Then, the transformation

$$u = r\cos(k\phi), \quad v = r\sin(k\phi) \qquad (\diamondsuit)$$

yields:

$$\ddot{u} + \omega^2 u = 0, \quad \ddot{v} + \omega^2 v = 0,$$

a two-dim harmonic oscillator in (u, v). If we apply the linearizing transformation $(\diamondsuit)$ to: $2i\psi_t + \psi_{uu} + \psi_{vv} - \omega^2(u^2 + v^2)\psi = 0$,

Harmonic oscillator on a double cone

G.Gubbiotti & MCN, J. Nonlinear Math.Phys., 2017

N.B.: $m = 1$, opening angle $2\alpha, \alpha \in \left(0, \frac{\pi}{2}\right), k = \sin(\alpha)$

The Lagrangian $L_{\mathsf{ho}} = \frac{1}{2}\left(\dot{r}^2 + k^2 r^2 \dot{\phi}^2\right) - \frac{1}{2}\omega^2 r^2$ admits 8 Noether symmetries, and the corresponding Lagrangian equations

$$\ddot{r} = k^2 r\dot{\phi}^2 - \omega^2 r, \quad \ddot{\phi} = -2\frac{\dot{r}\dot{\phi}}{r} \qquad (\spadesuit)$$

admit a 15-dim Lie symmetry algebra. Then, the transformation

$$u = r\cos(k\phi), \quad v = r\sin(k\phi) \qquad (\diamondsuit)$$

yields:

$$\ddot{u} + \omega^2 u = 0, \quad \ddot{v} + \omega^2 v = 0,$$

a two-dim harmonic oscillator in (u, v). If we apply the linearizing transformation $(\diamondsuit)$ to: $2\mathrm{i}\psi_t + \psi_{uu} + \psi_{vv} - \omega^2(u^2 + v^2)\psi = 0$, then the Schrödinger equation corresponding to system $(\spadesuit)$ is

$$2\mathrm{i}\psi_t + \psi_{rr} + \frac{1}{r}\psi_r + \frac{1}{k^2 r^2}\psi_{\phi\phi} - \omega^2 r^2\psi = 0.$$

Noether symmetries of the Lagrangian:

$$L_{\text{ho}} = \frac{1}{2}\left(\dot{r}^2 + k^2 r^2 \dot{\phi}^2\right) - \frac{1}{2}\omega^2 r^2$$

$\Gamma_8 = \partial_\phi$, $\Gamma_9 = \partial_t$, $\Gamma_{10} = \cos(2\omega t)\partial_t - \omega\sin(2\omega t)r\partial_r$,
$\Gamma_{11} = \sin(2\omega t)\partial_t + \omega\cos(2\omega t)r\partial_r$,
$\Gamma_{12} = \cos(\omega t)\left(\cos(k\phi)\partial_r - \frac{1}{kr}\sin(k\phi)\partial_\phi\right)$,
$\Gamma_{13} = \sin(\omega t)\left(\cos(k\phi)\partial_r - \frac{1}{kr}\sin(k\phi)\partial_\phi\right)$,
$\Gamma_{14} = \cos(\omega t)\left(\sin(k\phi)\partial_r + \frac{1}{kr}\cos(k\phi)\partial_\phi\right)$,
$\Gamma_{15} = \sin(\omega t)\left(\sin(k\phi)\partial_r + \frac{1}{kr}\cos(k\phi)\partial_\phi\right)$.

Noether symmetries of the Lagrangian:

$$L_{\text{ho}} = \tfrac{1}{2}\left(\dot{r}^2 + k^2 r^2 \dot{\phi}^2\right) - \tfrac{1}{2}\omega^2 r^2$$

$\Gamma_8 = \partial_\phi$, $\Gamma_9 = \partial_t$, $\Gamma_{10} = \cos(2\omega t)\partial_t - \omega\sin(2\omega t)r\partial_r$,
$\Gamma_{11} = \sin(2\omega t)\partial_t + \omega\cos(2\omega t)r\partial_r$,
$\Gamma_{12} = \cos(\omega t)\left(\cos(k\phi)\partial_r - \tfrac{1}{kr}\sin(k\phi)\partial_\phi\right)$,
$\Gamma_{13} = \sin(\omega t)\left(\cos(k\phi)\partial_r - \tfrac{1}{kr}\sin(k\phi)\partial_\phi\right)$,
$\Gamma_{14} = \cos(\omega t)\left(\sin(k\phi)\partial_r + \tfrac{1}{kr}\cos(k\phi)\partial_\phi\right)$,
$\Gamma_{15} = \sin(\omega t)\left(\sin(k\phi)\partial_r + \tfrac{1}{kr}\cos(k\phi)\partial_\phi\right)$.

Lie symmetries of the Schrödinger equation:

$$2\mathrm{i}\psi_t + \psi_{rr} + \frac{1}{r}\psi_r + \frac{1}{k^2 r^2}\psi_{\phi\phi} - \omega^2 r^2 \psi = 0$$

$\Omega_1 = \Gamma_8, \Omega_2 = \Gamma_9, \Omega_3 = \Gamma_{10} + \omega\left(\sin(2\omega t) - 2\mathrm{i}\cos(2\omega t)\omega r^2\right)\psi\partial_\psi$,
$\Omega_4 = \Gamma_{11} - \omega\left(\cos(2\omega t) + 2\mathrm{i}\sin(2\omega t)\omega r^2\right)\psi\partial_\psi$,
$\Omega_5 = \Gamma_{12} - \mathrm{i}\omega r\sin(\omega t)\cos(k\phi)\psi\partial_\psi$,
$\Omega_6 = \Gamma_{13} + \mathrm{i}\omega r\cos(\omega t)\cos(k\phi)\psi\partial_\psi$,
$\Omega_7 = \Gamma_{14} - \mathrm{i}\omega r\sin(\omega t)\sin(k\phi)\psi\partial_\psi$,
$\Omega_8 = \Gamma_{15} + \mathrm{i}\omega r\cos(\omega t)\cos(k\phi)\psi\partial_\psi$.

Who is right?

K. Kowalski, J. Rembieliński, Annals of Physics, 2013 took the usual angular momentum $\hat{p}_\phi = -\mathrm{i}\partial_\phi$ and looked for self-adjoint operators of the type $\hat{p}_r = -\mathrm{i}\,(\partial_r + F(r))$, with respect to the scalar product $\langle f, g \rangle = \int_0^{2\pi}\!\!\int_{-\infty}^{\infty} f^* g\,|r|\,\mathrm{d}r\mathrm{d}\phi$, on the space of square integrable functions $f(r,\phi), g(r,\phi)$ on the cone. This yields that the self-adjoint operator is $\hat{p}_r = -\mathrm{i}\left(\partial_r + \frac{1}{2r}\right)$, and consequently their Schrödinger equation is:

$$2\mathrm{i}\psi_t + \psi_{rr} + \frac{1}{r}\psi_r + \frac{1}{4k^2r^2}\psi_{\phi\phi} - \left(\frac{1}{4r^2} + \omega^2 r^2\right)\psi = 0.$$

The additional term $-\frac{\psi}{4r^2}$ breaks the symmetries, i.e. four out of eight symmetries are not preserved.

Also ($p \in \mathbb{Z}$):

$$E_n = \omega\left(2n + \frac{1}{2}\sqrt{1 + \frac{4p^2}{k^2}} + 1\right) \text{ vs. } E_n = \omega\left(2n + \frac{|p|}{k} + 1\right).$$

Further insight is needed especially from the experimentalists. However...

DeWitt's approach

Introducing the self-adjoint momentum operators as defined in
Bryce Seligman DeWitt, Phys. Rev., 1952, i.e.:

$$\hat{p}_k = -\mathrm{i}\left(\frac{\partial}{\partial_{q_k}} + \Gamma^j_{kj}\right)$$

with Γ^j_{jk} the contracted Christoffel symbols, yield the same angular
$\hat{p}_\phi = -\mathrm{i}\partial_\phi$ and radial momenta $\hat{p}_r = -\mathrm{i}\left(\partial_r + \frac{1}{2r}\right)$ given by K&R.

DeWitt's approach

Introducing the self-adjoint momentum operators as defined in *Bryce Seligman DeWitt, Phys. Rev., 1952*, i.e.:

$$\hat{p}_k = -\mathrm{i}\left(\frac{\partial}{\partial_{q_k}} + \Gamma^j_{kj}\right)$$

with Γ^j_{jk} the contracted Christoffel symbols, yield the same angular $\hat{p}_\phi = -\mathrm{i}\partial_\phi$ and radial momenta $\hat{p}_r = -\mathrm{i}\left(\partial_r + \frac{1}{2r}\right)$ given by K&R. However, the Hamiltonian operator shall include an extra quantum mechanical potential Q, *"which must be added to the covariant classical Hamiltonian in order to produce the covariant quantum Hamiltonian"* as stated by DeWitt.

Adding this potential Q means to eliminate the symmetry-breaking term $-\psi/4r^2$, and consequently

DeWitt's approach

Introducing the self-adjoint momentum operators as defined in *Bryce Seligman DeWitt, Phys. Rev., 1952*, i.e.:

$$\hat{p}_k = -\mathrm{i}\left(\frac{\partial}{\partial q_k} + \Gamma^j_{kj}\right)$$

with Γ^j_{jk} the contracted Christoffel symbols, yield the same angular $\hat{p}_\phi = -\mathrm{i}\partial_\phi$ and radial momenta $\hat{p}_r = -\mathrm{i}\left(\partial_r + \frac{1}{2r}\right)$ given by K&R. However, the Hamiltonian operator shall include an extra quantum mechanical potential Q, *"which must be added to the covariant classical Hamiltonian in order to produce the covariant quantum Hamiltonian"* as stated by DeWitt.
Adding this potential Q means to eliminate the symmetry-breaking term $-\psi/4r^2$, and consequently **the quantization method that preserves the Noether symmetries of the classical problem corresponds to DeWitt's approach.**

Noether vs Schrödinger

Applying Noether's theorem yields that L_{13}, L_{62} and L_{87} admit five Noether point symmetries. The main difference among the three Lagrangians is that they admit different Noether point symmetries. Which Schrödinger-type equations are obtained??:

$$L_{8,7} = \frac{1}{2}\dot{x}^2 \implies 2i\psi_t + \psi_{xx} = 0$$

Noether vs Schrödinger

Applying Noether's theorem yields that L_{13}, L_{62} and L_{87} admit five Noether point symmetries. The main difference among the three Lagrangians is that they admit different Noether point symmetries. Which Schrödinger-type equations are obtained??:

$$L_{8,7} = \frac{1}{2}\dot{x}^2 \implies 2i\psi_t + \psi_{xx} = 0$$

$$L_{6,2} = \frac{1}{2\dot{x}} \implies$$

Noether vs Schrödinger

Applying Noether's theorem yields that L_{13}, L_{62} and L_{87} admit five Noether point symmetries. The main difference among the three Lagrangians is that they admit different Noether point symmetries. Which Schrödinger-type equations are obtained??:

$$L_{8,7} = \frac{1}{2}\dot{x}^2 \implies 2i\psi_t + \psi_{xx} = 0$$

$$L_{6,2} = \frac{1}{2\dot{x}} \implies 2i\psi_x + \psi_{tt} = 0$$

Noether vs Schrödinger

Applying Noether's theorem yields that L_{13}, L_{62} and L_{87} admit five Noether point symmetries. The main difference among the three Lagrangians is that they admit different Noether point symmetries. Which Schrödinger-type equations are obtained??:

$$L_{8,7} = \frac{1}{2}\dot{x}^2 \implies 2i\psi_t + \psi_{xx} = 0$$

$$L_{6,2} = \frac{1}{2\dot{x}} \implies 2i\psi_x + \psi_{tt} = 0$$

$$L_{1,3} = \frac{1}{2t^2(x - t\dot{x})} \implies$$

Noether vs Schrödinger

Applying Noether's theorem yields that L_{13}, L_{62} and L_{87} admit five Noether point symmetries. The main difference among the three Lagrangians is that they admit different Noether point symmetries. Which Schrödinger-type equations are obtained??:

$$L_{8,7} = \frac{1}{2}\dot{x}^2 \quad \Longrightarrow \quad 2i\psi_t + \psi_{xx} = 0$$

$$L_{6,2} = \frac{1}{2\dot{x}} \quad \Longrightarrow \quad 2i\psi_x + \psi_{tt} = 0$$

$$L_{1,3} = \frac{1}{2t^2(x - t\dot{x})} \quad \Longrightarrow \quad t^2\psi_{tt} + 2tx\psi_{tx} + x^2\psi_{xx} = 0$$

$$\Longrightarrow$$

Noether vs Schrödinger

Applying Noether's theorem yields that L_{13}, L_{62} and L_{87} admit five Noether point symmetries. The main difference among the three Lagrangians is that they admit different Noether point symmetries. Which Schrödinger-type equations are obtained??:

$$L_{8,7} = \frac{1}{2}\dot{x}^2 \implies 2i\psi_t + \psi_{xx} = 0$$

$$L_{6,2} = \frac{1}{2\dot{x}} \implies 2i\psi_x + \psi_{tt} = 0$$

$$L_{1,3} = \frac{1}{2t^2(x - t\dot{x})} \implies t^2\psi_{tt} + 2tx\psi_{tx} + x^2\psi_{xx} = 0$$

$$\Rightarrow \xi = \frac{x}{t} \Rightarrow$$

Noether vs Schrödinger

Applying Noether's theorem yields that L_{13}, L_{62} and L_{87} admit five Noether point symmetries. The main difference among the three Lagrangians is that they admit different Noether point symmetries. Which Schrödinger-type equations are obtained??:

$$L_{8,7} = \frac{1}{2}\dot{x}^2 \implies 2i\psi_t + \psi_{xx} = 0$$

$$L_{6,2} = \frac{1}{2\dot{x}} \implies 2i\psi_x + \psi_{tt} = 0$$

$$L_{1,3} = \frac{1}{2t^2(x - t\dot{x})} \implies t^2\psi_{tt} + 2tx\psi_{tx} + x^2\psi_{xx} = 0$$

$$\Rightarrow \xi = \frac{x}{t} \Rightarrow \frac{\partial^2\psi}{\partial t^2}(t,\xi) = 0.$$

Noether vs Schrödinger

Applying Noether's theorem yields that L_{13}, L_{62} and L_{87} admit five Noether point symmetries. The main difference among the three Lagrangians is that they admit different Noether point symmetries. Which Schrödinger-type equations are obtained??:

$$L_{8,7} = \frac{1}{2}\dot{x}^2 \implies 2i\psi_t + \psi_{xx} = 0$$

$$L_{6,2} = \frac{1}{2\dot{x}} \implies 2i\psi_x + \psi_{tt} = 0$$

$$L_{1,3} = \frac{1}{2t^2(x - t\dot{x})} \implies t^2\psi_{tt} + 2tx\psi_{tx} + x^2\psi_{xx} = 0$$

$$\Rightarrow \xi = \frac{x}{t} \Rightarrow \frac{\partial^2 \psi}{\partial t^2}(t, \xi) = 0.$$

EITHER INVERT TIME WITH SPACE
OR GO BACK TO CLASSICAL MECHANICS !!!

Quantizing with Noether: Final Act (?)

Quantizing with Noether: Final Act (?)

- Find the Lie symmetries of the Lagrange equations

$$\Upsilon = W(t, \underline{x})\partial_t + \sum_{k=1}^{N} W_k(t, \underline{x})\partial_{x_k}$$

Quantizing with Noether: Final Act (?)

- Find the Lie symmetries of the Lagrange equations

$$\Upsilon = W(t, \underline{x})\partial_t + \sum_{k=1}^{N} W_k(t, \underline{x})\partial_{x_k}$$

- Find the right Noether symmetries

$$\Gamma = V(t, \underline{x})\partial_t + \sum_{k=1}^{N} V_k(t, \underline{x})\partial_{x_k}, \quad \Gamma \subset \Upsilon$$

Quantizing with Noether: Final Act (?)

- Find the Lie symmetries of the Lagrange equations

$$\Upsilon = W(t, \underline{x})\partial_t + \sum_{k=1}^{N} W_k(t, \underline{x})\partial_{x_k}$$

- Find the <u>right</u> Noether symmetries

$$\Gamma = V(t, \underline{x})\partial_t + \sum_{k=1}^{N} V_k(t, \underline{x})\partial_{x_k}, \quad \Gamma \subset \Upsilon$$

- Construct the Schrödinger equation admitting those Lie symmetries

$$2iu_t + \sum_{k,j=1}^{N} f_{kj}(\underline{x})u_{x_j x_k} + \sum_{k=1}^{N} h_k(\underline{x})u_{x_k} + f_3(\underline{x})u = 0$$

$$\Omega = V(t, \underline{x})\partial_t + \sum_{k=1}^{N} V_k(t, \underline{x})\partial_{x_k} + G(t, \underline{x}, u)\partial_u$$

Quantizing with Noether: Final Act (?)

- Find the Lie symmetries of the Lagrange equations

$$\Upsilon = W(t, \underline{x})\partial_t + \sum_{k=1}^{N} W_k(t, \underline{x})\partial_{x_k}$$

- Find the <u>right</u> Noether symmetries

$$\Gamma = V(t, \underline{x})\partial_t + \sum_{k=1}^{N} V_k(t, \underline{x})\partial_{x_k}, \quad \Gamma \subset \Upsilon$$

- Construct the Schrödinger equation admitting those Lie symmetries

$$2iu_t + \sum_{k,j=1}^{N} f_{kj}(\underline{x})u_{x_j x_k} + \sum_{k=1}^{N} h_k(\underline{x})u_{x_k} + f_3(\underline{x})u = 0$$

$$\Omega = V(t, \underline{x})\partial_t + \sum_{k=1}^{N} V_k(t, \underline{x})\partial_{x_k} + G(t, \underline{x}, u)\partial_u$$

QUANTIZE PRESERVING THE RIGHT SYMMETRIES!

Quantizing with Noether: Final Act (?)

- Find the Lie symmetries of the Lagrange equations

$$\Upsilon = W(t, \underline{x})\partial_t + \sum_{k=1}^{N} W_k(t, \underline{x})\partial_{x_k}$$

- Find the right Noether symmetries

$$\Gamma = V(t, \underline{x})\partial_t + \sum_{k=1}^{N} V_k(t, \underline{x})\partial_{x_k}, \quad \Gamma \subset \Upsilon$$

- Construct the Schrödinger equation admitting those Lie symmetries

$$2iu_t + \sum_{k,j=1}^{N} f_{kj}(\underline{x})u_{x_j x_k} + \sum_{k=1}^{N} h_k(\underline{x})u_{x_k} + f_3(\underline{x})u = 0$$

$$\Omega = V(t, \underline{x})\partial_t + \sum^{N} V_k(t, \underline{x})\partial_{x_k} + G(t, \underline{x}, u)\partial_u$$

QUANTIZE PRESERVING THE RIGHT REPRESENTATION!

Final Remarks

We have exemplified different instances where Noether's first theorem plays a fundamental role: finding conservation laws, going from classical to quantum mechanics, eliminating 'ghosts'. Also, we have shown that Noether symmetries are ubiquitous, namely they exist even for systems without a Lagrangian.

Final Remarks

We have exemplified different instances where Noether's first theorem plays a fundamental role: finding conservation laws, going from classical to quantum mechanics, eliminating 'ghosts'. Also, we have shown that Noether symmetries are ubiquitous, namely they exist even for systems without a Lagrangian. Of course, our examples are not exhaustive of the many fields where Noether's theorems can be used. In the last 100 years, many mathematicians, physicists, and mathematical physicists have applied Noether's theorems in their research. And in the next 100 years?

Final Remarks

We have exemplified different instances where Noether's first theorem plays a fundamental role: finding conservation laws, going from classical to quantum mechanics, eliminating 'ghosts'. Also, we have shown that Noether symmetries are ubiquitous, namely they exist even for systems without a Lagrangian. Of course, our examples are not exhaustive of the many fields where Noether's theorems can be used. In the last 100 years, many mathematicians, physicists, and mathematical physicists have applied Noether's theorems in their research. And in the next 100 years?

William Morris (1893): **The past is not dead, but is living in us, and will be alive in the future which we are helping to make.**