

FANTASTIC SYMMETRIES AND WHERE TO FIND THEM

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COLEGIADO DE FÍSICA
Universidade Estadual do Sudoeste da Bahia
November 12, 2020, Itapetinga, BA

Outline

- Lie symmetry method: a brief review with examples.
- Taming chaotic systems by means of Lie symmetry method: an example.
- The complete symmetry group: operators vs equations.
- Lagrangian equations and Noether's first theorem.
- Missed Lie and Noether symmetries: an example.
- Is Mechanics a branch of mathematical analysis?

The role of symmetry in fundamental physics

[David J. Gross, PNAS, 1996]

*Symmetry principles play an important role with respect to the laws of nature. They summarize the regularities of the laws that are independent of the specific dynamics. Thus invariance principles provide a structure and coherence to the laws of nature just as the laws of nature provide a structure and coherence to the set of events. Indeed, it is hard to imagine that much progress could have been made in deducing the laws of nature without the existence of certain **symmetries**.*

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Along with Frank Wilczek and David Politzer, he was awarded the 2004 Nobel Prize in Physics *for the discovery of asymptotic freedom in the theory of the strong interaction.*



Which symmetries?

"Historically, the applications of Lie groups to differential equations pioneered by Lie and Noether faded into obscurity just as the global, abstract reformulation of differential geometry by E. Cartan gained its ascendancy in the mathematical community"

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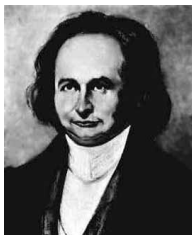
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while a senior theoretical physicist asks:

"What are these symmetries?"

Jacobi and Lie: their legacy



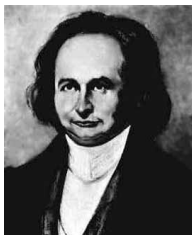
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Ordinary differential equations of first-order

$$\Gamma = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}$$

$$\text{First prolongation : } \Gamma_1 = \Gamma + \left(\frac{d\eta}{dx} - y' \frac{d\xi}{dx} \right) \frac{\partial}{\partial y'}$$

Γ generates a **Lie point symmetry** for equation:

$$H(x, y, y') \equiv y' - f(x, y) = 0$$

if and only if

$$\Gamma_1 (H(x, y, y')) \Big|_{H=0} = 0$$

This yields the following undetermined **determining equation**

$$\xi f_x + \eta f_y - \frac{\partial \eta}{\partial x} - \frac{\partial \eta}{\partial y} f + \frac{\partial \xi}{\partial x} f + \frac{\partial \xi}{\partial y} f^2 = 0$$

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Link between a **Lie symmetry** Γ and an **integrating factor** μ

$$\mu = \frac{1}{\eta - f\xi}$$

EXAMPLE of a first-order ODE

L. Euler, *Novi Commentarii academiae scientiarum Petropolitanae* 17 (1772) 105-124.

$$\frac{dy}{dx} = -nx^{n-2} - \frac{x^{2n-3}}{y}$$

Integrating factor:

$$\mu = (y + x^{n-1})^{-n}$$

Lie symmetry:

$$\Gamma = \frac{\partial}{\partial x} + \left[\frac{(y + x^{n-1})^n}{y} - \frac{nx^{n-2}y + x^{2n-3}}{y} \right] \frac{\partial}{\partial y}$$

A first-order ODE (or system) admits an **infinite-dimensional** Lie symmetry algebra.

INCREASING THE ORDER DECREASES THE DIMENSION!

Ordinary differential equations of second-order

$$\Gamma = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}$$

Second prolongation:

$$\Gamma_2 = \Gamma_1 + \left[\frac{d}{dx} \left(\frac{d\eta}{dx} - y' \frac{d\xi}{dx} \right) - y'' \frac{d\xi}{dx} \right] \frac{\partial}{\partial y''}$$

Γ is a **Lie point symmetry** of the ODE:

$$Eq(x, y, y', y'') \equiv y'' - f(x, y, y') = 0$$

if and only if

$$\Gamma_2 (Eq(x, y, y', y'')) \Big|_{Eq=0} = 0$$

This yields the following **determining equation**

$$\eta_{xy} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})(y')^2 + -(y')^3\xi_{yy} + (\eta_y - 2\xi_x - 3y'\xi_y)f - \xi f_x - \eta f_y - [\eta_x + (\eta_y - \xi_x)y' - (y')^2\xi_y] = 0$$

an **overdetermined** system of linear PDEs.

An EXAMPLE of a second-order ODE

F. Brauer, in *Mathematical Approaches for Emerg. and Reemerg. Inf. Diseases*, Springer, 2002, pp. 31-65.

$$yy'' - y'^2 + y^2y' + yy' + y^3 + y^2 = 0 \quad (*)$$

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Then (*) is reduced to the following first-order **autonomous** ODE

$$\frac{dY}{dX} + Y + 1 = 0 \Rightarrow (Y + 1)e^X = B \Rightarrow \frac{y'}{y} + 1 + y = Be^{-x}$$

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$$y = \frac{Be^{-x}}{A \exp [Be^{-x}] + 1}$$

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In order to solve it we transform it into a canonical form. We consider the two-dimensional Abelian intransitive subalgebra of the Lie symmetry algebra $2A_2$, i.e.

$$X_1 = \Gamma_1 - 3C_2\Gamma_3, \quad X_2 = \Gamma_2.$$

The corresponding canonical transformations are (see Lie's book):

$$\tilde{y} = \frac{u + y}{3(uy - C_2)}, \quad \tilde{u} = -\frac{1}{uy - 3C_2}.$$

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From a butterfly to a tornado with symmetries

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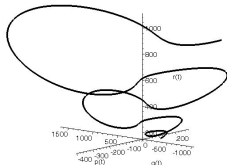
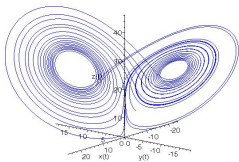
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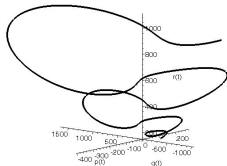
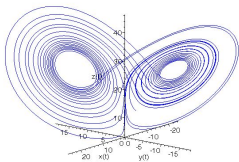
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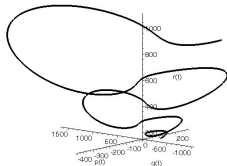
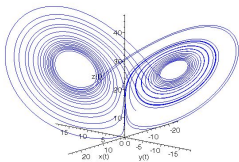
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Lie symmetries can transform a butterfly into a tornado...

From a butterfly to a tornado with symmetries

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Lie symmetries can transform a butterfly into a tornado... and a tornado into a butterfly [MCN, J. Math. Phys. 44 (2003)].

Lorenz system (1963)

$$\begin{cases} x' &= \tilde{\sigma}(y - x) \\ y' &= -xz + \tilde{r}x - y \\ z' &= xy - \tilde{b}z \end{cases}$$

Segur's integrable case (1980):

$$\begin{cases} x' &= \frac{(y-x)}{2} \\ y' &= -xz - y \\ z' &= xy - z \end{cases}$$

corresponding third order ODE:

$$2xx''' - 2x'x'' + 5xx'' - 3x'^2 + 2x^3x' + 3xx' + x^4 + x^2 = 0$$

Lie symmetry algebra L_2 :

$$X_1 = \partial_\tau, \quad X_2 = e^{\tau/2} \left(\partial_\tau - \frac{x}{2} \partial_x \right)$$

Euler-Poinsot system (1750)

$$\begin{cases} \dot{p} = \frac{(B-C)}{A}qr \\ \dot{q} = \frac{(C-A)}{B}pr \\ \dot{r} = \frac{(A-B)}{C}pq \end{cases}$$

corresponding third order ODE:

$$p \ddot{\ddot{p}} - \dot{p} \ddot{\ddot{p}} - \frac{4(C-A)(A-B)}{BC} p^3 \dot{p} = 0$$

Lie symmetry algebra \mathcal{L}_2 :

$$\Gamma_1 = \partial_t, \quad \Gamma_2 = t\partial_t - p\partial_p$$

L_2 and \mathcal{L}_2 are the same, i.e., Type IV in Lie's classification.

Transformation from EPS to LIS:

$$\begin{cases} \tau = \log\left(\frac{4}{t^2}\right) \\ x = \frac{pt}{2} \\ y = \frac{C-B}{2A}qrt^2 \\ z = \frac{C-B}{2A} \left[\frac{(C-A)}{B}r^2 + \frac{(A-B)}{C}q^2 \right] t^2 \end{cases}$$

and assuming:

$$B = \frac{4A(A - C)}{4A - 3C}$$

Transformation from EPS to LIS:

$$\left\{ \begin{array}{l} \tau = \log\left(\frac{4}{t^2}\right) \\ x = \frac{pt}{2} \\ y = -\frac{(2A-C)(2A-3C)}{2A(4A-3C)}qrt^2 \\ z = -\frac{(2A-C)(2A-3C)[4A^2q^2-(4A-3C)^2r^2]}{8A^2(4A-3C)^2}t^2 \end{array} \right.$$

Transformation from LIS to EPS:

$$\left\{ \begin{array}{l} t = 2e^{-\tau/2} \\ p = xe^{\tau/2} \\ q = -e^{\tau/2} \frac{(4A-3C)y}{2\sqrt{(2A-C)(2A-3C)(\sqrt{y^2+z^2}+z)}} \\ r = Ae^{\tau/2} \frac{\sqrt{\sqrt{y^2+z^2}+z}}{\sqrt{(2A-C)(2A-3C)}} \end{array} \right.$$

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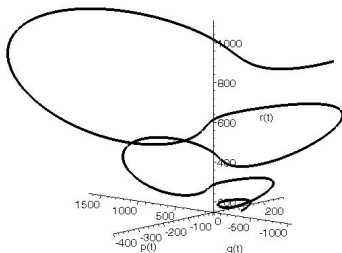
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What happens if one applies the above transformation to the general Lorenz system?

$$\left\{ \begin{array}{l} \dot{p} = \frac{2(2A-C)(2A-3C)\tilde{\sigma}}{A(4A-3C)}qr + (2\tilde{\sigma} - 1)\frac{p}{t} \\ \dot{q} = \frac{3C-4A}{4A}pr + (\tilde{b} - 1)\frac{4A^2q^2 - (4A-3C)^2r^2}{4A^2q^2 + (4A-3C)^2r^2}\frac{q}{t} \\ \quad + \tilde{r}\frac{2(4A-3C)^3A}{(2A-C)(2A-3C)[4A^2q^2 + (4A-3C)^2r^2]}\frac{pr}{t^2} \\ \dot{r} = \frac{A}{4A-3C}pq - (\tilde{b} - 1)\frac{4A^2q^2 - (4A-3C)^2r^2}{4A^2q^2 + (4A-3C)^2r^2}\frac{r}{t} \\ \quad + \tilde{r}\frac{8(4A-3C)A^3}{(2A-C)(2A-3C)[4A^2q^2 + (4A-3C)^2r^2]}\frac{pq}{t^2} \end{array} \right.$$

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A **tornado** appears!



Complete symmetry group

In [Krause, J.Math.Phys. 35 (1994)] an extended notion of *symmetry in mechanics* was introduced, to characterize a *classical system by the symmetry laws it obeys*, namely *different mechanical systems cannot have exactly the same symmetry properties*.

Nonlocal symmetries were considered:

$$Y = \left[\int \xi(t, x_1, \dots, x_N) dt \right] \partial_t + \sum_{k=1}^N \eta_k(t, x_1, \dots, x_N) \partial_{x_k},$$

and then applied to Kepler's problem:

$$Y_1 = 2 \left(\int x_1 dt \right) \partial_t + x_1^2 \partial_{x_1} + x_1 x_2 \partial_{x_2} + x_1 x_3 \partial_{x_3}$$

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In [MCN, J.Math.Phys.37, 1996] it was shown that they can be retrieved by applying Lie group analysis to the equivalent nonautonomous systems.

Complete symmetry group of the Riccati chain

In [Muriel & Romero, NARWA (2014)] it was shown that the nonlocal **complete symmetry group** of each member of the Riccati chain can be given by means of λ -symmetries.

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Let us begin with the first-order Riccati equation, i.e.

$$\dot{u} = -u^2. \quad (1)$$

Among the infinite number of Lie symmetries, there are the following two:

$$\begin{aligned} \Gamma_1 &= u(u\partial_u), \\ \Gamma_2 &= u(tu - 1)\partial_u, \end{aligned}$$

which form a two-dimensional non-Abelian intransitive Lie algebra (Lie Type IV). It is easy to show that those two symmetries represent the complete symmetry group of equation (1).

Next let us consider the second-order Riccati equation, and write it as a first-order system, i.e.

$$\begin{cases} \dot{u}_1 = u_2, \\ \dot{u}_2 = -(3u_1u_2 + u_1^3). \end{cases} \quad (2)$$

Among the infinite number of Lie symmetries, there are the following three:

$$\Gamma_1 = R_1(u_1\partial_{u_1} + (D - u_1)(u_1)\partial_{u_2}),$$

$$\Gamma_2 = R_1((tu_1 - 1)\partial_{u_1} + (D - u_1)(tu_1 - 1)\partial_{u_2}),$$

$$\Gamma_3 = R_1((t^2u_1 - 2t)\partial_{u_1} + (D - u_1)(t^2u_1 - 2t)\partial_{u_2}),$$

where $D = \partial_t + u_2\partial_{u_1}$ and $R_1 = u_2 + u_1^2$. Those three symmetries generate **the complete symmetry group** of system (2).

The third-order equation in the Riccati chain can be written as the following first-order system:

$$\begin{cases} \dot{u}_1 &= u_2, \\ \dot{u}_2 &= u_3, \\ \dot{u}_3 &= -(4u_1u_3 + 3u_2^2 + 6u_1^2u_2 + u_1^4). \end{cases} \quad (3)$$

Among the infinite number of Lie symmetries, there are the following four:

$$\begin{aligned} \Gamma_1 &= R_2(u_1\partial_{u_1} + (D - u_1)(u_1)\partial_{u_2} + (D - u_1)^2(u_1)\partial_{u_3}), \\ \Gamma_2 &= R_2((tu_1 - 1)\partial_{u_1} + (D - u_1)(tu_1 - 1)\partial_{u_2} + (D - u_1)^2(tu_1 - 1)\partial_{u_3}), \\ \Gamma_3 &= R_2((t^2u_1 - 2t)\partial_{u_1} + (D - u_1)(t^2u_1 - 2t)\partial_{u_2} + (D - u_1)^2(t^2u_1 - 2t)\partial_{u_3}), \\ \Gamma_4 &= R_2((t^3u_1 - 3t^2)\partial_{u_1} + (D - u_1)(t^3u_1 - 3t^2)\partial_{u_2} + (D - u_1)^2(t^3u_1 - 3t^2)\partial_{u_3}), \end{aligned}$$

where $D = \partial_t + u_2\partial_{u_1} + u_3\partial_{u_2}$, and $R_2 = u_3 + 3u_1u_2 + u_1^3$. Those four symmetries generate **the complete symmetry group** of system (3).

Consequently, the complete symmetry group of the first-order system corresponding to the n th equation of the Riccati chain, R_n , is generated by the following $n + 1$ operators:

$$\Gamma_1 = R_{n-1} \sum_{j=0}^{n-1} (D - u_1)^j (u_1) \partial_{u_{j+1}},$$

$$\Gamma_1 = R_{n-1} \sum_{j=0}^{n-1} (D - u_1)^j (tu_1 - 1) \partial_{u_{j+1}},$$

⋮

$$\Gamma_{n+1} = R_{n-1} \sum_{j=0}^{n-1} (D - u_1)^j (t^n u_1 - nt^{n-1}) \partial_{u_{j+1}},$$

where $D = \partial_t + \sum_{j=1}^{n-1} u_{j+1} \partial_{u_j}$, and R_{n-1} is the $(n - 1)$ member of the Riccati chain.

Why not the complete symmetry group

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$$X_1 = \partial_t, \quad X_2 = \left(t - \frac{1}{u} \right) \partial_t,$$

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However, those two symmetries **DO NOT** represent the complete symmetry group since they are also two symmetries of

$$\dot{u} = 0.$$

About 102 years ago

Invariante Variationsprobleme.

(F. Klein zum fünfzigjährigen Doktorjubiläum.)

Von

Emmy Noether in Göttingen.

Vorgelegt von F. Klein in der Sitzung vom 26. Juli 1918¹⁾.

Es handelt sich um Variationsprobleme, die eine kontinuierliche Gruppe (im Lieschen Sinne) gestatten; die daraus sich ergebenden Folgerungen für die zugehörigen Differentialgleichungen finden ihren allgemeinsten Ausdruck in den in § 1 formulierten, in den folgenden Paragraphen bewiesenen Sätzen. Über diese aus Variationsproblemen entspringenden Differentialgleichungen lassen sich viel präzisere Aussagen machen als über beliebige, eine Gruppe gestattende Differentialgleichungen, die den Gegenstand der Lieschen Untersuchungen bilden. Das folgende beruht also auf einer Verbindung der Methoden der formalen Variationsrechnung mit denen der Lieschen Gruppentheorie. Für spezielle Gruppen und Variationsprobleme ist diese Verbindung der Methoden nicht neu; ich erwähne Hamel und Herglotz für spezielle endliche, Lorentz und seine Schüler (z. B. Fokker), Weyl und Klein für spezielle unendliche Gruppen²⁾. Insbesondere sind die zweite Kleinsche Note und die vorliegenden Ausführungen gegenseitig durch einander beein-

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For those differential equations that arise from variational problems, the statements that can be formulated are much more precise than for the arbitrary differential equations that are invariant under a group, which are the subject of Lie's researches. What follows thus depends upon a combination of the methods of the formal calculus of variations and of Lie's theory of groups.

Noether's Theorems



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Here we are mainly concern with **Noether's First Theorem**.

Lagrangian equations and Noether's first theorem

Variational problem most familiar to physicists:

$$L = L(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)),$$

with its corresponding (Euler)-Lagrangian equations:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) = \frac{\partial L}{\partial q_k}, \quad (k = 1, \dots, n).$$

Then a Noether symmetry has to satisfy the following relationship:

$$L \frac{d\xi}{dt} + \Gamma_1(L) = \frac{df}{dt},$$

where $f = f(t, \mathbf{q})$ is a function to be determined, and Γ_1 is the first prolongation of $\Gamma = \xi(t, \mathbf{q})\partial_t + \sum_{k=1}^n \eta_k(t, \mathbf{q})\partial_{q_k}$.

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$$I = \xi L + \sum_{k=1}^n \frac{\partial L}{\partial \dot{q}_k} (\eta_k - \dot{q}_k \xi) - f = \text{const.}$$

Missing Noether symmetries

The key to find Noether symmetries is **the boundary term f** . In Mechanics courses, students are usually taught **very simple Noether symmetries** of the natural Lagrangian (namely, Kinetic minus Potential energy), e.g., translation on time, that yield $f = \text{constant}$. Indeed, one may have to deal with **a very complicated f** in order to find a Noether symmetry.

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Fang et al, *Phys. Lett. A*, 2010 presented the following Lagrangian

$$L = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2}k(q_1^2 + q_2^2) + tq_1, \quad (k, m = \text{const}), \quad (4)$$

and its corresponding Lagrangian equations

$$\ddot{q}_1 = -\frac{k}{m}q_1 + \frac{t}{m}, \quad \ddot{q}_2 = -\frac{k}{m}q_2. \quad (5)$$

There **three first integrals** were determined, with the claim that **only one** was related to Noether symmetries.

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There **three first integrals** were determined, with the claim that **only one** was related to Noether symmetries.

In **MCN, Phys. Lett. A, 2011**, **eight Noether symmetries** and corresponding conserved quantities were derived.

System (5) is linear therefore it admits a **fifteen-dimensional Lie point symmetry algebra**, isomorphic to $sl(4, \mathbf{R})$. Then, the Lagrangian (4) admits **eight Noether symmetries** and corresponding eight conserved quantities:

$$\Gamma_1 - \Gamma_5 \Rightarrow I_1 = kq_1 \dot{q}_2 - t \dot{q}_2 + q_2 - kq_2 \dot{q}_1$$

$$\Gamma_6 \Rightarrow I_6 = m \dot{q}_2 \cos \left(\sqrt{\frac{k}{m}} t \right) + q_2 \sqrt{\frac{m}{k}} \sin \left(\sqrt{\frac{k}{m}} t \right)$$

$$\Gamma_7 \Rightarrow I_7 = -q_2 \sqrt{\frac{m}{k}} \cos \left(\sqrt{\frac{k}{m}} t \right) + m \dot{q}_2 \sin \left(\sqrt{\frac{k}{m}} t \right)$$

$$\begin{aligned} \Gamma_8 \Rightarrow I_8 = & -\sqrt{\frac{m}{k}} \left(k^3 (q_1^2 + q_2^2) - m + k (2m\dot{q}_1 + t^2) \right. \\ & \left. - k^2 (m (\dot{q}_1^2 + \dot{q}_2^2) + 2tq_1) \right) \left(1 - 2 \sin^2 \left(\sqrt{\frac{k}{m}} t \right) \right) \\ & - 2km \left(k(q_1 + \dot{q}_1 t) - t - k^2(q_1 \dot{q}_1 + q_2 \dot{q}_2) \right) \sin \left(2\sqrt{\frac{k}{m}} t \right) \end{aligned}$$

$$\begin{aligned} \Gamma_9 \Rightarrow I_9 = & \sqrt{\frac{m}{k}} \left(k^3 (q_1^2 + q_2^2) - m + k (2m\dot{q}_1 + t^2) \right. \\ & \left. - k^2 (m (\dot{q}_1^2 + \dot{q}_2^2) + 2tq_1) \right) \sin \left(\sqrt{\frac{k}{m}} t \right) \cos \left(\sqrt{\frac{k}{m}} t \right) \\ & + km \left(k(q_1 + \dot{q}_1 t) - t - k^2(q_1 \dot{q}_1 + q_2 \dot{q}_2) \right) \left(1 - 2 \sin^2 \left(\sqrt{\frac{k}{m}} t \right) \right) \end{aligned}$$

$$\begin{aligned} \Gamma_{10} &\Rightarrow l_{10} = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2) + \frac{1}{2}k(q_1^2 + q_2^2) - tq_1 - \frac{m}{k}\dot{q}_1 + \frac{t^2}{2k} \\ \Gamma_{14} &\Rightarrow l_{14} = \sqrt{\frac{m}{k}}(kq_1 - t) \sin\left(\sqrt{\frac{k}{m}}t\right) + (k\dot{q}_1 - 1) \cos\left(\sqrt{\frac{k}{m}}t\right) m \\ \Gamma_{15} &\Rightarrow l_{15} = \sqrt{\frac{m}{k}}(kq_1 - t) \cos\left(\sqrt{\frac{k}{m}}t\right) - (k\dot{q}_1 - 1) \sin\left(\sqrt{\frac{k}{m}}t\right) m. \end{aligned}$$

None admits $f = \text{constant}$, e.g.:

$$\begin{aligned} \Gamma_6 &\Rightarrow f = -q_2\sqrt{mk} \sin\left(\sqrt{\frac{k}{m}}t\right), \\ \Gamma_8 &\Rightarrow f = 2tq_1 - \frac{t^2}{2k} + \frac{m}{k^2} - k(q_1^2 + q_2^2) \\ &\quad - 2\sqrt{\frac{m}{k}}(kq_1 - t) \cos\left(\sqrt{\frac{k}{m}}t\right) \sin\left(\sqrt{\frac{k}{m}}t\right) \\ &\quad - \left(4tq_1 - \frac{t^2}{k} + \frac{m}{k^2} - 2k(q_1^2 + q_2^2)\right) \sin^2\left(\sqrt{\frac{k}{m}}t\right). \end{aligned} \tag{6}$$

Further details on system (5) in [MCN, Phys. Lett. A, 2011](#).

Lagrange vindicated



In the Avertissement to his "*Mécanique Analytique*" (1788) Joseph-Louis Lagrange (1736-1813) wrote:

The methods that I explain in it require neither constructions nor geometrical or mechanical arguments, but only the algebraic operations inherent to a regular and uniform process. *Those who love Analysis will, with joy, see mechanics become a new branch of it and will be grateful to me for thus having extended its field.* (tr. by J.R. Maddox:)

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It is a joke, isn't it??!!

Final remarks

David J. Gross, PNAS, 1996

*We are embarked on a new stage of exploration of fundamental laws of nature, a voyage guided largely by the search for and the discovery of new **symmetries**.*



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In the mean time, let's search for these hidden symmetries...

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Fantastic symmetries and where to find them

