VII CICLO DE SEMINÁRIOS DE FÍSICA
Itapetinga 2021

Colegiado de Física da Universidade Estadual do Sudoeste da Bahia

UESB - ITAPETINGA - BA

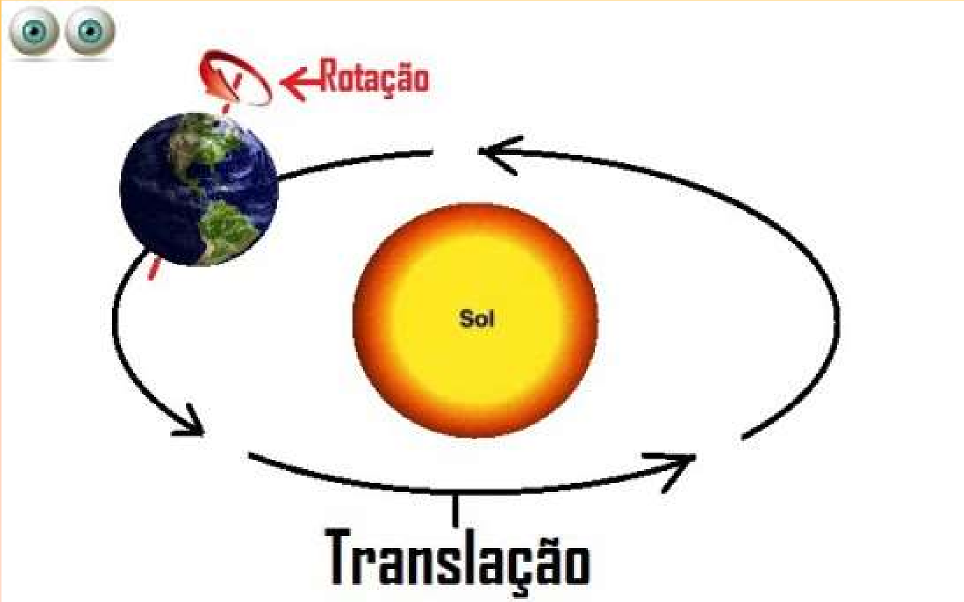
Equação de Dirac e Spinores na Frente de Luz

Prof. Titular: Jorge Henrique Sales
Discentes: Diego Ramos, Gabriel Aragão.
Universidade Estadual de Santa Cruz-UESC
Contato: jhosales@uesc.br

Universidade Estadual de Santa Cruz - UESC

Conteúdo

O que é Spin?



Fonte Google

- O que é um spin?
  - Atributo intrínseco da partícula;
  - Uma analogia de uma grandeza clássica para o spin é o momento angular da partícula;

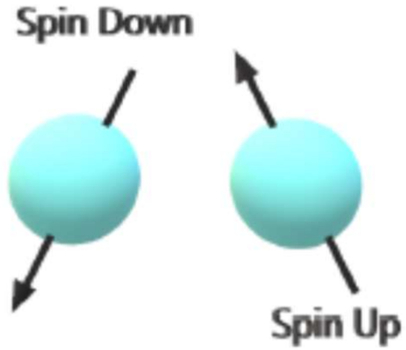


Figure 1.1: Estados do spin do elétron

- A equação de Dirac para o elétron livre em sua forma covariante é dada por

$$\left(i\gamma^{\mu}\partial_{\mu} - m\hat{1}\right)\varphi(\bar{x}) = 0 \qquad (1.1)$$

onde $\mu = 0, 1, 2, 3$ e:

$$\varphi(\bar{x}) = \begin{bmatrix} \varphi_1(\bar{x}) \\ \varphi_2(\bar{x}) \\ \varphi_3(\bar{x}) \\ \varphi_4(\bar{x}) \end{bmatrix}$$

$$\gamma^{\mu} = \left(\gamma^{0}, \hat{\vec{\gamma}}\right) \tag{1.2}$$

$$\gamma^{0} = \begin{pmatrix} \hat{1} & 0 \\ 0 & -\hat{1} \end{pmatrix} \tag{1.3}$$

$$\hat{\vec{\gamma}} = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{1.4}$$

com $\mu = \{0, 1, 2, 3\}$ e $i = \{1, 2, 3\}$.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Supondo que a solução de $\varphi(\bar{x})$ seja do tipo onda plana, isto é:

$$\varphi(\bar{x}) = \exp(-i\bar{p} \cdot \bar{x})u(\bar{p}) \tag{1.5}$$

$$u(\bar{p}) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$\bar{x} = (x^0, \vec{r})$$

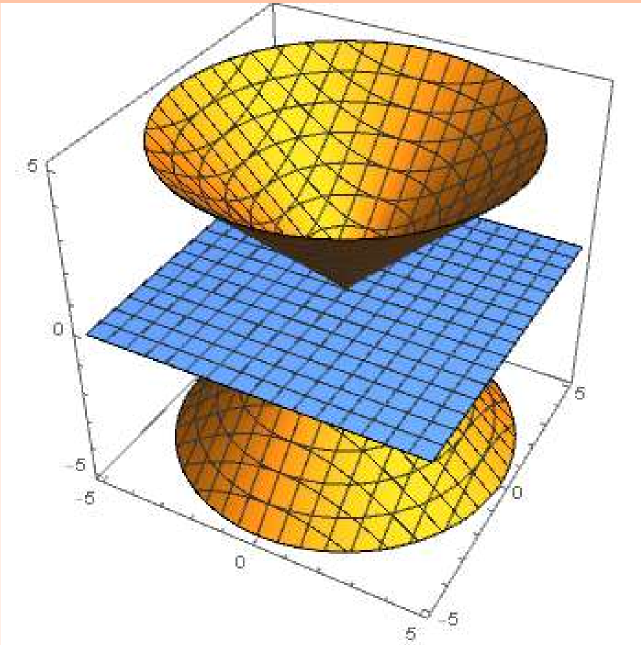
$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

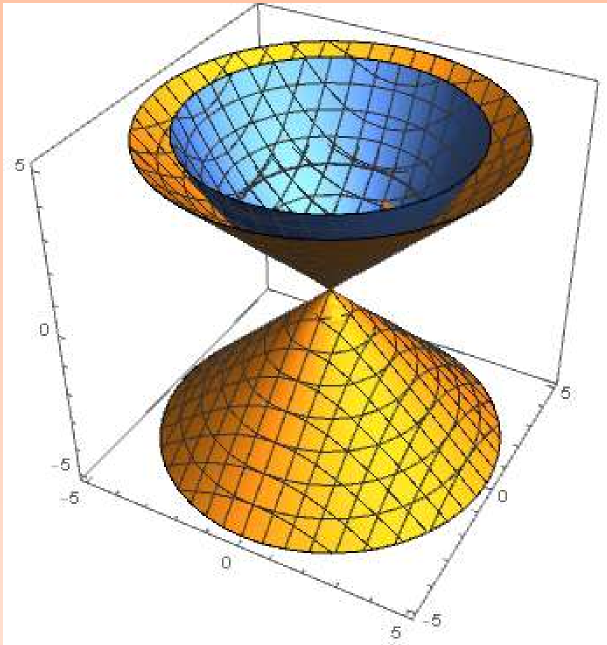
$$\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

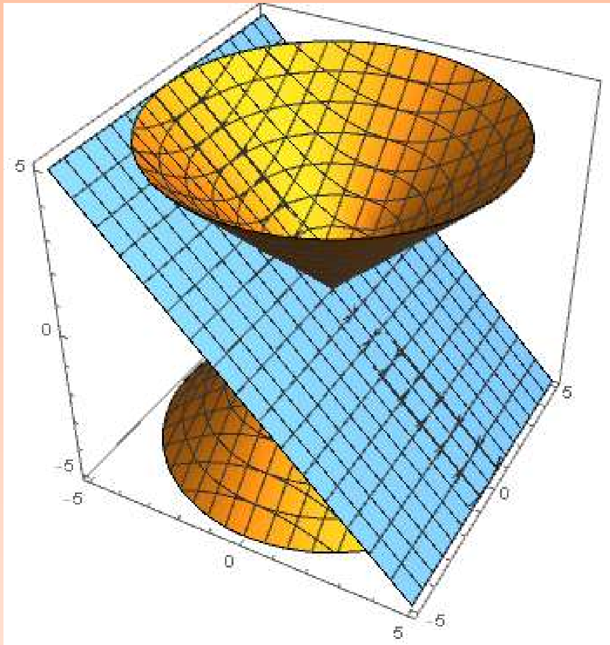
$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

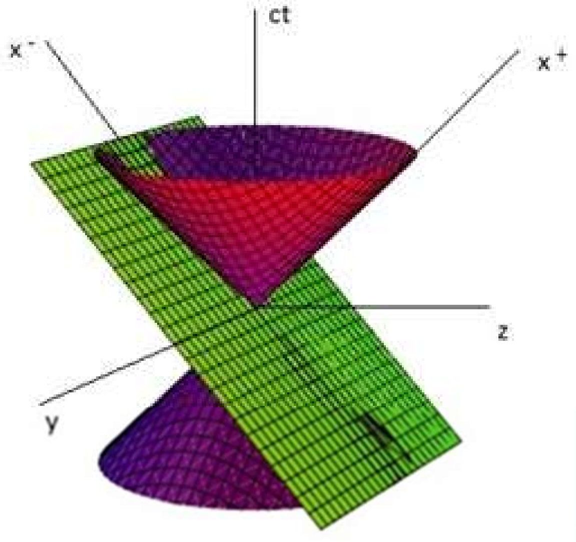
$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

Coordenadas da Frente de luz









$$\begin{cases} x^+ & = \frac{2}{\sqrt{2}}(x^0 + x^3) \\ x^- & = \frac{\sqrt{2}}{2}(x^0 - x^3) \\ \vec{x}^\perp & = x^1\vec{i} + x^2\vec{j} \end{cases}$$

$$\begin{cases} p^+ & = \frac{\sqrt{2}}{2}(p^0 + p^3) \\ p^- & = \frac{\sqrt{2}}{2}(p^0 - p^3) \\ p^\perp & = p^1\vec{i} + p^2\vec{j} \end{cases}$$

$$ds^2 = 2dx^+dx^- - (d\mathbf{x}^\perp)^2.$$

$$k^\mu k_\mu = m^2 \,,$$

$$k^\mu k_\mu = 2k^+k^- - \vec{k}^\perp \vec{k}^\perp \,.$$

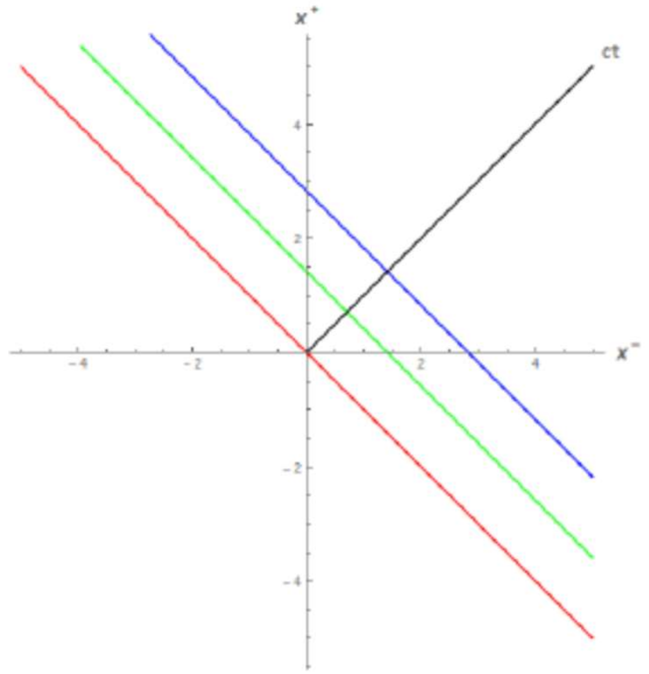


Figura 2.2: Frente de Luz em duas dimensões.

$$k^- = \frac{\vec{k}_\perp^2 + m^2}{2k^+} \,.$$

Equação de Dirac na Frente de Luz

$$\left(i\gamma^{\mu}\partial_{\mu} - m\hat{1}\right)\varphi(\bar{x}) = 0$$

$$\begin{cases} \gamma^{+} &= \frac{\sqrt{2}}{2}(\gamma^{0} + \gamma^{3}) \\ \gamma^{-} &= \frac{\sqrt{2}}{2}(\gamma^{0} - \gamma^{3}) \\ \gamma^{\perp} &= \begin{bmatrix} \gamma^{1} & \gamma^{2} \end{bmatrix} \end{cases}$$

$$\partial_0 = \frac{\partial}{\partial x^0} = \frac{\partial x^+}{\partial x^0}\frac{\partial}{\partial x^+} + \frac{\partial x^1}{\partial x^0}\frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial x^0}\frac{\partial}{\partial x^2} + \frac{\partial x^-}{\partial x^0}\frac{\partial}{\partial x^-} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x^+} + \frac{\partial}{\partial x^-}\right)$$

$$\partial_3 = \frac{\partial}{\partial x^3} = \frac{\partial x^+}{\partial x^3}\frac{\partial}{\partial x^+} + \frac{\partial x^1}{\partial x^3}\frac{\partial}{\partial x^1} + \frac{\partial x^2}{\partial x^3}\frac{\partial}{\partial x^2} + \frac{\partial x^-}{\partial x^3}\frac{\partial}{\partial x^-} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x^+} - \frac{\partial}{\partial x^-}\right)$$

$$\partial^0 = \frac{\partial}{\partial x_0} = \frac{\partial x_+}{\partial x_0}\frac{\partial}{\partial x_+} + \frac{\partial x_1}{\partial x_0}\frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial x_0}\frac{\partial}{\partial x_2} + \frac{\partial x_-}{\partial x_0}\frac{\partial}{\partial x_-} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_+} + \frac{\partial}{\partial x_-}\right)$$

$$\partial^3 = \frac{\partial}{\partial x_3} = \frac{\partial x_+}{\partial x_3}\frac{\partial}{\partial x_+} + \frac{\partial x_1}{\partial x_3}\frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial x_3}\frac{\partial}{\partial x_2} + \frac{\partial x_-}{\partial x_3}\frac{\partial}{\partial x_-} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_+} - \frac{\partial}{\partial x_-}\right)$$

$$\Psi_{\mathrm{LF}}(x^+, x^\perp, x^-) = e^{i(k^- x^+ + k^+ x^- - k^\perp x^\perp)}$$

$$\left[(\gamma^+ p^- + \gamma^- p^+) - \hat{\gamma}^\perp \hat{p}^\perp - m\hat{1}\right] u = 0$$

$$\gamma^+ = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

$$\gamma^- = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}$$

$$\left[\left(\frac{\gamma^+ + \gamma^-}{\sqrt{2}}\right)\left(\frac{p^+ + p^-}{\sqrt{2}}\right) - \left(\frac{\gamma^+ - \gamma^-}{\sqrt{2}}\right)\left(\frac{p^+ - p^-}{\sqrt{2}}\right) - \hat{\gamma}^\perp \hat{p}^\perp - m\hat{1}\right] u = 0$$

$$\left(\frac{\gamma^+ p^+ + \gamma^+ p^- + \gamma^- p^+ + \gamma^- p^-}{2} - \frac{\gamma^+ p^+ - \gamma^+ p^- - \gamma^- p^+ + \gamma^- p^-}{2} - \hat{\gamma}^\perp \hat{p}^\perp - m\hat{1}\right) u = 0$$

$$\begin{pmatrix} \frac{p^- + p^+}{\sqrt{2}} - m & 0 & \frac{p^- - p^+}{\sqrt{2}} & -p^1 + ip^2 \\ 0 & \frac{p^- + p^+}{\sqrt{2}} - m & -p^1 - ip^2 & \frac{-p^- + p^+}{\sqrt{2}} \\ \frac{-p^- + p^+}{\sqrt{2}} & p^1 - ip^2 & \frac{-p^- - p^+}{\sqrt{2}} - m & 0 \\ p^1 + ip^2 & \frac{p^- - p^+}{\sqrt{2}} & 0 & \frac{-p^- - p^+}{\sqrt{2}} - m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} \frac{p^- + p^+}{\sqrt{2}} - m & 0 & \frac{p^- - p^+}{\sqrt{2}} & -p^1 + ip^2 \\ 0 & \frac{p^- + p^+}{\sqrt{2}} - m & -p^1 - ip^2 & \frac{-p^- + p^+}{\sqrt{2}} \\ \frac{-p^- + p^+}{\sqrt{2}} & p^1 - ip^2 & \frac{-p^- - p^+}{\sqrt{2}} - m & 0 \\ p^1 + ip^2 & \frac{p^- - p^+}{\sqrt{2}} & 0 & \frac{-p^- - p^+}{\sqrt{2}} - m \end{vmatrix} = 0$$

Para resolver o determinante vamos utilizar a *regra de Laplace*, apresentada abeixo:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$A_{ij} = (-1)^{i+j} \cdot D_{ij}$$

$$4{p^-}^2{p^+}^2+m^4-4p^-p^+m^2+{p^1}^4-4p^-p^+{p^1}^2+2m^2{p^1}^2+{p^2}^4-4p^-p^+{p^2}^2+2m^2{p^2}^2+2{p^1}^2{p^2}^2=0$$

$$p^-=\frac{{p^\perp}^2+m^2}{2p^+}$$

Caso: $p^- > 0$

(a) Spin up: ↑

Para essa situação, vamos convencionar que:

$$u_1 = 1$$

$$u_2 = 0$$

substituindo nas equações do sistema (15), temos:

$$\begin{cases} \left(\frac{p^- + p^+}{\sqrt{2}}\right) + \left(\frac{p^- + p^+}{\sqrt{2}}\right) u_3 + (ip^\perp - p^\perp) u_4 = 0 \\ (-ip^1 + p^2) u_3 + \left(\frac{-p^- + p^+}{\sqrt{2}}\right) u_4 = 0 \\ \left(\frac{-p^- + p^+}{\sqrt{2}}\right) - \left(\frac{p^- + p^+}{\sqrt{2}} + m\right) u_3 = 0 \\ (ip^2 + p^1) - \left(\frac{p^- + p^+}{\sqrt{2}} + m\right) u_4 = 0 \end{cases}$$

$$\mathbf{u}'_\uparrow = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \frac{-p^- + p^+}{p^- + p^+ + \sqrt{2}m} \\ \frac{\sqrt{2}(p^1 + ip^2)}{p^- + p^+ + \sqrt{2}m} \end{bmatrix}$$

Spin down: $\downarrow$

Tomemos

$$u_1 = 0$$

$$u_2 = 1$$

substituindo no sistema (15), teremos:

$$\begin{cases} \left(\frac{p^- - p^+}{\sqrt{2}}\right) u_3 + (-p^1 + ip^2)\, u_4 = 0 \\ \left(\frac{p^- + p^+}{\sqrt{2}} - m\right) + (-p^1 - ip^2)\, u_3 + \left(\frac{-p^- + p^+}{\sqrt{2}}\right) u_4 = 0 \\ (p^1 - ip^2) + \left(\frac{-p^- - p^+}{\sqrt{2}} - m\right) u_3 = 0 \\ \left(\frac{p^- - p^+}{\sqrt{2}}\right) + \left(\frac{-p^- - p^+}{\sqrt{2}} - m\right) u_4 = 0 \end{cases}$$

$$\mathbf{u}''_{\downarrow} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{\sqrt{2}(p^1 - ip^2)}{p^- + p^+ + \sqrt{2}m} \\ \frac{p^- - p^+}{p^- + p^+ + \sqrt{2}m} \end{bmatrix}$$

Caso $p^+ < 0$

Nesse caso, a equação (11), toma a forma:

$$(\hat{\gamma}^+\hat{p}^- - \hat{\gamma}^-\hat{p}^+ - \hat{\gamma}^\perp\hat{p}^\perp - m\hat{1})u = 0 \rightarrow$$

$$\begin{pmatrix} \frac{(p^- - p^+)}{\sqrt{2}} - m & 0 & \frac{(p^- + p^+)}{\sqrt{2}} & -p^1 + ip^2 \\ 0 & \frac{(p^- - p^+)}{\sqrt{2}} - m & -p^1 - ip^2 & \frac{(-p^- - p^+)}{\sqrt{2}} \\ \frac{(-p^- - p^+)}{\sqrt{2}} & p^1 - ip^2 & \frac{(-p^- + p^+)}{\sqrt{2}} - m & 0 \\ p^1 + ip^2 & \frac{(p^- + p^+)}{\sqrt{2}} & 0 & \frac{(-p^- + p^+)}{\sqrt{2}} - m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} \frac{(p^- - p^+)}{\sqrt{2}} - m & 0 & \frac{(p^- + p^+)}{\sqrt{2}} & -p^1 + ip^2 \\ 0 & \frac{(p^- - p^+)}{\sqrt{2}} - m & -p^1 - ip^2 & \frac{(-p^- - p^+)}{\sqrt{2}} \\ \frac{(-p^- - p^+)}{\sqrt{2}} & p^1 - ip^2 & \frac{(-p^- + p^+)}{\sqrt{2}} - m & 0 \\ p^1 + ip^2 & \frac{(p^- + p^+)}{\sqrt{2}} & 0 & \frac{(-p^- + p^+)}{\sqrt{2}} - m \end{vmatrix} = 0$$

$$\frac{1}{16}(64{p^+}^2{p^-}^2 + 64p^+p^-{p^1}^2 + 16{p^1}^4 + 64p^+p^-{p^2}^2 + 32{p^1}^2{p^2}^2 + 16{p^2}^4 + 64p^+p^-m^2 + 32{p^1}^2m^2$$
$$+ 32{p^2}^2m^2 + 16m^4) = 0$$

$$(2p^+p^- + {p^1}^2 + {p^2}^2 + m^2)^2 = 0$$

$$(2p^+p^- + {p^\perp}^2 + m^2)^2 = 0$$

$$2p^+p^- + {p^\perp}^2 + m^2 = 0$$

$$p^- = -\frac{{p^\perp}^2 + m^2}{2p^+}$$

$$\begin{cases} \left(\frac{p^- - p^+}{\sqrt{2}} - m\right) u_1 + 0u_2 + \left(\frac{p^- + p^+}{\sqrt{2}}\right) u_3 + (-p^1 + ip^2)\, u_4 = 0 \\ 0u_1 + \left(\frac{p^- - p^+}{\sqrt{2}} - m\right) u_2 + (-p^1 - ip^2)\, u_3 + \left(\frac{-p^- - p^+}{\sqrt{2}}\right) u_4 = 0 \\ \left(\frac{-p^- - p^+}{\sqrt{2}}\right) u_1 + (p^1 - ip^2)\, u_2 + \left(\frac{-p^- + p^+}{\sqrt{2}} - m\right) u_3 + 0u_4 = 0 \\ (p^1 + ip^2)\, u_1 + \left(\frac{p^- + p^+}{\sqrt{2}}\right) u_2 + 0u_3 + \left(\frac{-p^- + p^+}{\sqrt{2}} - m\right) u_4 = 0 \end{cases}$$

spin up: ↑

Para essa situação, vamos convencionar que:

$u_3 = 1$

$u_4 = 0$

$$\mathbf{u}_\uparrow''' = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} \frac{-p^- - p^+}{p^- - p^+ - \sqrt{2}m} \\ \frac{\sqrt{2}(p^1 + ip^2)}{p^- - p^+ - \sqrt{2}m} \\ 1 \\ 0 \end{bmatrix}$$

spin down: ↓

$u_3 = 0$

$u_4 = 1$

$$\mathbf{u}_\downarrow'''' = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}(p^1 - ip^2)}{p^- - p^+ - \sqrt{2}m} \\ \frac{p^- + p^+}{p^- - p^+ - \sqrt{2}m} \\ 0 \\ 1 \end{bmatrix}$$

	Energia positiva		Energia negativa	
	spin up ↑	spin down ↓	spin up ↑	spin down ↓
u_1	1	0	$\frac{-p^- - p^+}{p^- - p^+ - \sqrt{2}m}$	$\frac{\sqrt{2}(p^1 - ip^2)}{p^- - p^+ - \sqrt{2}m}$
u_2	0	1	$\frac{\sqrt{2}(p^1 + ip^2)}{p^- - p^+ - \sqrt{2}m}$	$\frac{p^- + p^+}{p^- - p^+ - \sqrt{2}m}$
u_3	$\frac{-p^- + p^+}{p^- + p^+ + \sqrt{2}m}$	$\frac{\sqrt{2}(p^1 - ip^2)}{p^- + p^+ + \sqrt{2}m}$	1	0
u_4	$\frac{\sqrt{2}(p^1 + ip^2)}{p^- + p^+ + \sqrt{2}m}$	$\frac{p^- - p^+}{p^- + p^+ + \sqrt{2}m}$	0	1

	$E > 0$ (Espaço Minkowisk)		$p^+ > 0$ (Frente de Luz)	
	Spin up↑	Spin down ↓	Spin up ↑	Spin down ↓
u_1	1	0	1	0
u_2	0	1	0	1
u_3	$\frac{\hat{p}_z}{E+m}$	$\frac{\hat{p}_x - i\hat{p}_y}{E+m}$	$\frac{2(p^+)^2 - p_\perp^2 - m^2}{2(p^+)^2 + p_\perp^2 + m^2 + 2\sqrt{2}mp^+}$	$\frac{2\sqrt{2}p^+(p^1 - ip^2)}{2(p^+)^2 + p_\perp^2 + m^2 + 2\sqrt{2}p^+m}$
u_4	$\frac{\hat{p}_x + i\hat{p}_y}{E+m}$	$-\frac{\hat{p}_z}{E+m}$	$\frac{2\sqrt{2}p^+(p^1 + ip^2)}{2(p^+)^2 + p_\perp^2 + m^2 + 2\sqrt{2}mp^+}$	$\frac{p_\perp^2 + m^2 - 2(p^+)^2}{2(p^+)^2 + p_\perp^2 + m^2 + 2\sqrt{2}p^+m}$

	$E < 0$ (Espaço Minkowisk)		$p^+ < 0$ (Frente de Luz)	
	Spin up↑	Spin down ↓	Spin up ↑	Spin down ↓
u_1	$\frac{-\hat{p}_z}{E+m}$	$-\frac{\hat{p}_x - i\hat{p}_y}{E+m}$	$\frac{2(p^+)^2 - p_\perp^2 - m^2}{2(p^+)^2 + p_\perp^2 + m^2 + 2\sqrt{2}mp^+}$	$\frac{-2\sqrt{2}p^+(p^1 - ip^2)}{2(p^+)^2 + p_\perp^2 + m^2 + 2\sqrt{2}p^+m}$
u_2	$-\frac{\hat{p}_x + i\hat{p}_y}{E+m}$	$\frac{\hat{p}_z}{E+m}$	$\frac{-2\sqrt{2}p^+(p^1 + ip^2)}{2(p^+)^2 + p_\perp^2 + m^2 + 2\sqrt{2}mp^+}$	$\frac{p_\perp^2 + m^2 - 2(p^+)^2}{2(p^+)^2 + p_\perp^2 + m^2 + 2\sqrt{2}p^+m}$
u_3	1	0	1	0
u_4	0	1	0	1

OBSERVAÇÃO

Referencial próprio

$$P^{\mu} = (P^0, 0, 0, 0)$$

$$p^0 = E$$

$$\vec{p} = (0, 0, 0)$$

Frente de Luz → Referencial próprio

$$p^{\mu} = (p^-, p^+, p^{\perp})$$

$$\vec{p}^{\perp} = p^1\vec{i} + p^2\vec{j}$$

$$\vec{p}^{\perp} = 0$$

$$2p^+p^- = {p^{\perp}}^2 + m^2$$

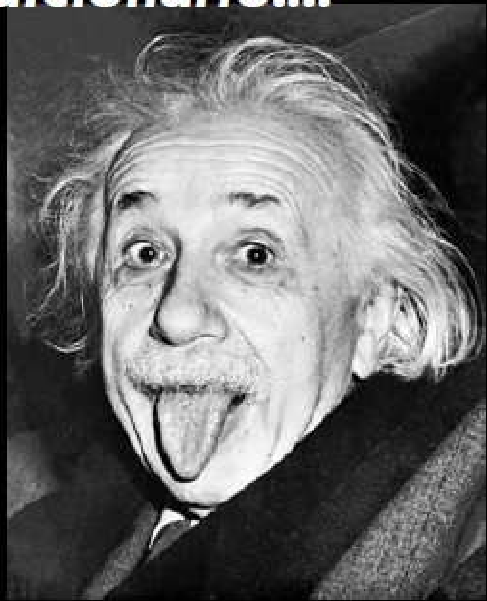
$$2p^+p^- = m^2$$

$$p^3 = \frac{p^+ - p^-}{\sqrt{2}} = 0, p^+ = p^- \rightarrow (p^+)^2$$

$$2(p^+)^2 = {p^{\perp}}^2 + m^2$$

- BASSALO, F, M, J. *Eletrodinâmica Quântica*, Livro v.1 2a.Ed, São Paulo, Livraria da Física 2006.
- MELO, João. P. B. C. *Covariância na Frente de Luz*, Tese de Doutorado, Universidade de São paulo Instituto de Física, 1996
- Sales, J. H.; A. T. Suzuk; L. A. Soriano. *Partículas e Antipartículas no Cone de Luz* Universidade Estadual de Santa Cruz, Ilhéus, BA, Brasil.
- Sales, J. H, et al.*Transformações de Lorentz na frente de luz*. Universidade Estadual de Santa Cruz, Departamento de Ciências Exatas e Tecnológicas, Ilhéus, BA, Brasil.

"O único lugar onde o sucesso vem antes do trabalho é no dicionário.!!!"



Tenham uma boa semana !!!

Agradecimentos

CNPq
Conselho Nacional de Desenvolvimento Científico e Tecnológico

fapesb
Fundação de Amparo à Pesquisa do Estado da Bahia

PPGMC

IN ALTUM