Dirac Bergmann Algorithm: An Overview

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Plan of The Talk

- Introduction
- Dirac Bergmann Theory of Constrained Systems
- First and Second Class Constraints
- Dirac Bracket and Its Properties
- Simple Example: Particle Constrained to a Spherical Surface
- Conclusion

Introduction

- Canonical quantization in a theory consists of obtaining the Poisson bracket relations between any two physical variables and carrying them over to the quantum commutation or anticommuatation relation with an $i\hbar$ prescription, namely

$$[A_{op}, B_{op}]_{\pm} = i\hbar\{A, B\} \tag{1}$$

- This method works very well in quantum mechanics. For example

$$[q_{op}, p_{op}]_{\pm} = i\hbar\{q, p\} = i\hbar \tag{2}$$

- However, in physical theories of quantum fields which often contain constraints, this method leads to inconsistencies.
- A quick way of observing this inconsistency is to suppose that there exists a constraint in our theory given by

$$\Gamma(q, p) = 0 \tag{3}$$

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- According to our prescription, in passage to the quantum theory, this must map to the null operator.

$$\Gamma(q,p) \to \phi_{op} \tag{4}$$

- It follows, therefore, that

$$[A_{op}, \Gamma_{op}]_{\pm} = [A_{op}, \phi_{op}]_{\pm} = i\hbar\{A, \Gamma\} \tag{5}$$

- The left hand side of this expression (eqn(5)) clearly vanishes as it is the commutator or anti-commutator of an operator with the null operator. The classical Poisson bracket on the right hand side, however is not in general zero. So, there is an inconsistency.
- So, in the case of constrained systems, the naive quantization procedure has to be modified.

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- Dirac recognised that the consistent way to quantize such a theory is to modify the naive Poisson brackets such that the new brackets (known as the Dirac Brackets) between a physical variable and a constraint vanish.
- Consequently one can write the quantum relations as

$$[A_{op}, B_{op}]_{\pm} = i\hbar\{A, B\}_D \tag{6}$$

and the consistency is overcome.

- Dirac and Bergman has given a detailed and systematic way of handling constrained systems which we are going to discuss in the next section.

- Let us consider a Lagrangian

$$L = L(q_i, \dot{q}_i) \quad \mathrm{i} = 1, 2, ..., \mathrm{N}, \tag{7}$$

where q_i and $\dot{q}_i$ represent N coordinates and velocities.

- The canonical momenta is defined as

$$p^i = \frac{\partial L}{\partial \dot{q}_i}, \tag{8}$$

- If now we want to go over Hamiltonian formalism we are looking at the transformation

$$(q_i, \dot{q}_i) \rightarrow (q_i, p^i). \tag{9}$$

- The Jacobian of this transformation is defined by the matrix

$$\frac{\partial p^i}{\partial \dot{q}_j} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}. \tag{10}$$

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- If the matrix in eqn(10) is nonsingular, then the transformation is unique and the naive canonical procedure goes through.
- However, in the most physical cases in quantum field theory, the matrix is singular. Consequently the transformation and hence the Hamiltonian for the system become nonunique.
- If we now analyze the eqn(10), then

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right) = 0. \tag{11}$$

  which implies that some of the momenta are not independent variables.
- Let the rank of the matrix $(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j})$ be R where $R < N$.

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- Then we can solve for R velocities as

$$\dot{q}^a = f^a(q, p^b, \dot{q}^\alpha) \quad \mathrm{a, b = 1, ..., R;} \quad \alpha = \mathrm{R + 1, ..., N.} \tag{12}$$

- Reinserting this into the definition of momenta, we obtain

$$p_i = \frac{\partial L}{\partial \dot{q}_i}(q, f^a, \dot{q}^\alpha) = g_i(q, p^a, \dot{q}^\alpha). \tag{13}$$

- For $a = 1, ..., R$

$$p_a = g_a. \tag{14}$$

- But for $\alpha = R + 1, ..., N$ one can show that

$$p_\alpha = g_\alpha(q, p^a) \quad \mathrm{a = 1, ..., R} \tag{15}$$

Cont...

- Thus we define

$$\Gamma_\alpha = p_\alpha - g_\alpha(q, p^a) \quad \alpha = \mathrm{R}+1, ..., \mathrm{N}; \quad \mathrm{a} = 1, ..., \mathrm{R} \tag{16}$$

- This can be written as

$$\Gamma_\alpha = 0 \quad \alpha = \mathrm{R}+1, ..., \mathrm{N} \tag{17}$$

  which are called the primary constraints.
- Let us define a $2N - (N - R) = N + R$ dimensional hypersurface Γ_c in the phase space Γ.
- We call two functions A, B on Γ weakly equal, $A \approx B$, if they are equal on Γ_c, namely

$$(A - B)|_{\Gamma_c} = 0 \tag{18}$$

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- Let us define the canonical Hamiltonian H_c as

$$H_c(q, p^a, \dot{q}^\alpha) = \sum_{b=1}^{R} p_b f^b(q, p^a, \dot{q}^\alpha) + \sum_{\alpha=R+1}^{N} \dot{q}_\alpha g_\alpha(q, p^a) - L(q, f^a, \dot{q}^\alpha). \tag{19}$$

where a and α are defined above.

- It has the property

$$\frac{\partial H_c}{\partial \dot{q}^\alpha} = 0 \quad \alpha = \mathrm{R} + 1, ..., \mathrm{N}. \tag{20}$$

- This implies that $H_c = H_c(q, p^a)$.
- We can write the equations of motion as

$$\dot{q}^i \approx \frac{\partial}{\partial p_i}(H_c + \sum_{\alpha=R+1}^{N} \dot{q}^\alpha \Gamma_\alpha), \quad \frac{\partial L}{\partial q^i} \approx -\frac{\partial}{\partial q_i}(H_c + \sum_{\alpha=R+1}^{N} \dot{q}^\alpha \Gamma_\alpha). \tag{21}$$

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- For solutions of Euler-Lagrange equation $\dot{p}^i = \frac{\partial L}{\partial q^i}$.
- Therefore we have

$$\dot{q}^i \approx \{q^i, H_c + \dot{q}^\alpha \Gamma_\alpha\}, \quad \dot{p}_i \approx \{p^i, H_c + \dot{q}^\alpha \Gamma_\alpha\}. \tag{22}$$

- The $\dot{q}^\alpha$ remain undetermined (since their Heisenberg equations of motions reduce to identities). We shall now on denote them by λ_α.
- The primary Hamiltonian can be written as

$$H_p = H_c + \lambda^\alpha \Gamma_\alpha \tag{23}$$

  where λ_α is undetermined.
- Here H_p contains only the $m = N - R$ primary constraints Γ_α.

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- Now the time evolution of any phase space function $A(q,p)$ not explicitly dependent on time is written as

$$\dot{A}(q,p) \approx \{A(q,p), H_c\} \tag{24}$$

- Furthermore, we want the constraints to have no dynamical evolution which requires

$$\dot{\Gamma}_\alpha \approx \{\Gamma_\alpha, H_p\} \approx \{\Gamma_\alpha, H_c\} + \lambda_\beta \{\Gamma_\alpha, \Gamma_\beta\} \approx 0 \tag{25}$$

- The above equation may either determine some of the unknown Lagrange multipliers or give rise to more functional relations between momenta and coordinates known as secondary constraints.
- One continues this process until all the constraints are determined to be evolution free.

First and Second Class Constraints

- Let us say, at this point that the total number of constraints in the system is n, $n < 2N$, and are given by

$$\Gamma_\alpha \approx 0 \quad \alpha = 1, 2..., \mathrm{n}. \tag{26}$$

- It is also evident that

$$\{\Gamma_\alpha, H_p\} \approx 0 \tag{27}$$

- Now, we will divide these constraints into two classes.
- Those constraints which have weakly vanishing Poisson brackets with every other constraint are called first class constraints, $\psi_\alpha, \alpha = 1, 2, ..., n_1$.
- Those which have atleast one nonvanishing Poisson bracket with the other constraints are known as second class, $\phi_\alpha, \alpha = 1, ..., n_2$; such that $n_1 + n_2 = n$.

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- The first class constraints ψ_α are associated with the local gauge invariances and one chooses gauge fixing conditions γ_β as additional constraints such that the first class constraints become second class.
- So, after gauge fixing all constraints become second class.
- Dirac has shown that they must be even in number.
- Let us denote them now collectively as

$$\Gamma_\alpha \approx 0 \quad 1, 2...2\mathrm{p}, 2\mathrm{p} < \mathrm{N}. \tag{28}$$

- Since these are all second class constraints, one can define the matrix of their Poisson brackets as

$$C_{\alpha\beta} \approx \{\Gamma_\alpha, \Gamma_\beta\}. \tag{29}$$

- The matrix $C_{\alpha\beta}$ is antisymmetric and Dirac has shown that it is nonsingular so that its inverse $C^{-1}_{\alpha\beta}$ exists.

Dirac Bracket and Its Properties

- Now, we will define a modified Poisson bracket (Dirac bracket) between two variables A and B as

$$\{A, B\}_D = \{A, B\} - \{A, \Gamma_\alpha\} C^{-1}_{\alpha\beta} \{\Gamma_\beta, B\}. \tag{30}$$

- Note that the Dirac bracket is defined such that any variable has a weakly vanishing Dirac bracket with any constraint, i.e.,

$$\begin{aligned} \{A, \Gamma_\alpha\}_D &= \{A, \Gamma_\alpha\} - \{A, \Gamma_\beta\} C^{-1}_{\beta\gamma} \{\Gamma_\gamma, \Gamma_\alpha\} \\ &\approx \{A, \Gamma_\alpha\} - \{A, \Gamma_\beta\} C^{-1}_{\beta\gamma} C_{\gamma\alpha} = \{A, \Gamma_\alpha\} - \{A, \Gamma_\alpha\} = 0 \end{aligned} \tag{31}$$

- It can be shown that the Dirac bracket is indeed the Poisson bracket when evaluated subject to the constraints in eqn(28).
- It is now straightforward to go over to quantum theory using

$$[A_{op}, B_{op}]_\pm = i\hbar \{A, B\}_D. \tag{32}$$

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- A very useful property of the Dirac brackets is the iterative property that is if there are a large number of constraints, one doesn't have to invert a large matrix but rather one can focus on a subset of all the constraints and define an intermediate Dirac bracket and so on.
- Secondly, although we have worked out everything for a finite dimension, the method extends readily to continuum field theory.
- One must, however, recognize that integration over intermediate variables is implied in relations such as eqn(30) and appropriate boundary conditions might be required.

Simple Example: Particle Constrained to a Spherical Surface

- Lagrangian for a particle constrained to a spherical surface is written as

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz + \frac{\lambda}{2}(x^2 + y^2 + z^2 - R^2) \tag{33}$$

where last term is the Lagrange multiplier term inforcing the constraint.

- The conjugate momenta are given by

$$p_x = m\dot{x}, \quad p_y = m\dot{y}, \quad p_z = m\dot{z}, \quad p_\lambda = 0 \tag{34}$$

- Now, the Hamiltonian for this system can be written as

$$H = \frac{p^2}{2m} + p_\lambda \dot{\lambda} + +mgz - \frac{\lambda}{2}(r^2 - R^2) \tag{35}$$

- The primary constraint for this system is

$$\Phi_1 = p_\lambda = 0 \tag{36}$$

- Including the primary constraint is Hamiltonian, the total Hamiltonian will be written as

$$H_T = \frac{p^2}{2m} + mgz - \frac{\lambda}{2}(r^2 - R^2) + up_\lambda \tag{37}$$

- The evolution of primary constraint with Hamiltonian H_T will give us secondary constraint which is written as

$$\Phi_2 = \{p_\lambda, H_T\} = r^2 - R^2 \tag{38}$$

- Now the evolution of secondary constraint will give us tertiary constraint

$$\Phi_3 = \{r^2 - R^2, H_T\} = p \cdot r \tag{39}$$

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- Similarly the quaternary constraint is written as

$$\Phi_4 = \{p \cdot r, H_T\} = p^2 - (mg - \lambda r)r \tag{40}$$

- The evolution of Φ_4 will determine u.
- The Poisson bracket between the constraints can be written as

$$\begin{aligned} \{\Phi_1, \Phi_4\}_P &= -r^2 \\ \{\Phi_2, \Phi_3\}_P &= r^2 \\ \{\Phi_2, \Phi_4\}_P &= p \cdot r \\ \{\Phi_3, \Phi_4\}_P &= p^2 + (mg - \lambda)r \end{aligned} \tag{41}$$

- The matrix between the constraints is written as

$$\{\Phi_a, \Phi_b\}_P = \begin{bmatrix} 0 & 0 & 0 & -r^2 \\ 0 & 0 & r^2 & p \cdot r \\ 0 & -r^2 & 0 & p^2 + (mg - \lambda)r \\ r^2 & -p \cdot r & -p^2 - (mg - \lambda)r & 0 \end{bmatrix} \tag{42}$$

- We can easily calculate the inverse of this matrix.
- Using the expression for Dirac bracket (eqn30) we can easily calculate the Dirac Brackets for this system which can be written as

$$\begin{aligned} \{r^a, p_b\}_D &= \delta^a_b - r^2, \\ \{r^a, r^b\}_D &= 0, \\ \{p_a, p_b\}_D &= (r^b p_a - r^a p_b) \end{aligned} \tag{43}$$

Conclusion

- The usual quantization procedure of quantum system doesn't work for a constrained system.
- To quantize the constrained system we follow a different formulation called Dirac-Bergmann algorithm for constrained system.
- Using this algorithm we analyze constrained of the system.
- These constrained can be distinguished into two classes called first and second class constraints.
- Using the algebra between the constraints a modified Poisson bracket named as Dirac bracket is constructed.
- Using the Dirac Bracket between the phase space variables we can construct equivalent commutator or anticommutator and hence canonically quantize the system.

Thank You