O Cálculo Fracionário e Algumas Aplicações

C. Godinho

November 11, 2021

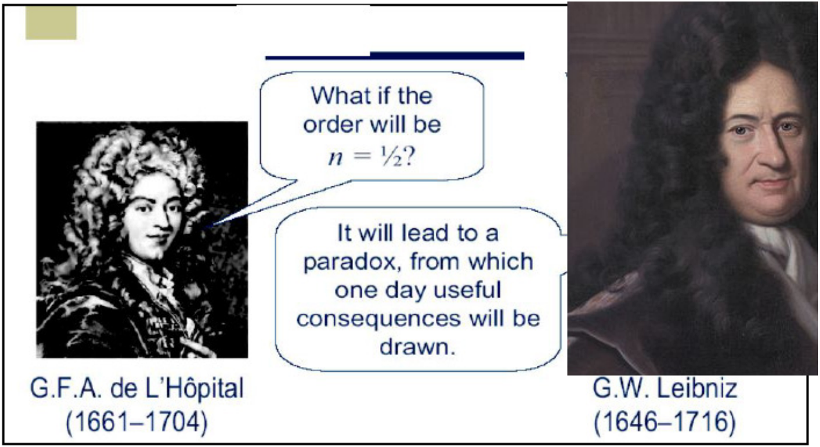
Índice

Origens do Cálculo

Newton: 1666-1669

Leibiniz:1675-1676

$\frac{d^n y}{dx^n}$ =? L'Hôpital [1695]



L.Euler [1730]

"Quando n for um inteiro positivo e f seja alguma função de x, a razão $d^n f/dx^n$ pode sempre ser expressa algebricamente. Isso pode ser facilmente entendido no caso inteiro, contudo não é evidente no caso se n é uma fração. Contudo com auxílio de extrapolação acredito que o tema possa ser analisado."

⋆ Lagrange [1772]

Indiretamente ajudou com sua lei dos expoentes

$$\boxed{\frac{d^m}{dx^m} \cdot \frac{d^n}{dx^n} = \frac{d^{m+n}}{dx^{m+n}}} \tag{1}$$

- Mais modernamente não usa-se do ponto, pois não estamos fazendo multiplicação aqui.
- Mais tarde, outros matemáticos ficaram intrigados na generalização desta regra para expoentes arbitrários.

Lacroix [1819]

Em 1819 Lacroix expressou a $n-$ésima derivada para uma função do tipo $y = x^m$ em que $(n \leq m)$:

$$\frac{d^n y}{dx^n} = \frac{m!}{(m-n)} x^{m-n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n}. \tag{2}$$

O que permitiu escrever para $m = 1, n = 1/2$,

$$\frac{d^{1/2} y}{dx^{1/2}} = \frac{2\sqrt{x}}{\sqrt{\pi}}. \tag{3}$$

onde :

$$\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} dt \tag{4}$$

Fourier [1822]

O próximo a fazer menção a derivadas fracionais foi Joseph Fourier em 1822. Sua definição para operações dessa natureza envolviam a representação integral de $f(x)$.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(a)da \int_{-\infty}^{\infty} cos\, p(x-a)dp\,, \tag{5}$$

e como, para n inteiro:

$$\frac{d^n}{dx^n} cosp(x-a) = p^n cos\left[p(x-a) + \frac{n\pi}{2}\right] \tag{6}$$

trocando para $n = u$ fracionário

$$\frac{d^u}{dx^u} f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(a)da \int_{-\infty}^{\infty} p^u cos\left[p(x-a) + \frac{u\pi}{2}\right] dp \tag{7}$$

Abel [1823]

Abel foi o primeiro que utilizou o formalismo para resolver uma equação integral que surge na formulação do problema da tautócrona

$$k = \int_0^x (x-t)^{-1/2} f(t)dt\,. \tag{8}$$

Abel estudou equações integrais com núcleos do tipo $(x-t)^\alpha$. Tal equação, exceto pelo fator multiplicativo $1/\Gamma(1/2)$, é um caso particular de uma integral definida que estabelece a integração fracional de ordem $1/2$. Abel escreveu o lado direito da equação como:

$$\sqrt{\pi}[d^{-1/2}/dx^{-1/2}]f(x), \tag{9}$$

então ele opera ambos os lados com $d^{-1/2}/dx^{-1/2}$ para obter

$$\frac{d^{1/2}}{dx^{1/2}}k = \sqrt{\pi}f(x), \tag{10}$$

onde fica claro que por causa da propriedade $D^{1/2}D^{-1/2}f = D^0f = f$. E ao calcular a derivada fracional de k determinamos $f(x)$, ressalva que nem sempre a derivada fracional de uma constante é zero.

Liouville [1832]

Ponto de partida: [1]

$$D^m e^{ax} = a^m e^{ax} \ , m \in \mathbb{N} \, , \tag{11}$$

derivadas de ordem arbitrária

$$D^\nu e^{ax} = a^\nu e^{ax} \ , \nu \in \mathbb{Q} \tag{12}$$

Idéia:

$$\boxed{f(x) = \sum_{n=0}^{\infty} c_n \, e^{a_n x}} \, , Re \ a_n > 0$$

$$D^\nu f(x) = \sum_{n=0}^{\infty} c_n a_n^\nu \, e^{a_n x} \, . \tag{13}$$

[1] The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order K.B. Oldham and J. Spanier (1974)

Liouville Segunda Abordagem

nessa abordagem ele considerou a integral

$$
\begin{aligned}
I &= \int_0^{\infty} u^{a-1} e^{-xu} du \;, \boxed{x = ut} \\
I &= x^{-a} \int_0^{\infty} t^{a-1} e^{-t} dt \\
x^{-a} &= \frac{I}{\Gamma(a)} \\
D^{\nu} x^{-a} &= \frac{(-1)^{\nu}}{\Gamma(a)} \int_0^{\infty} u^{a+\nu-1} e^{-xu} du \\
D^{\nu} x^{-a} &= \frac{(-1)^{\nu} \Gamma(a+\nu)}{\Gamma(a)} x^{-a-\nu} \qquad (14)
\end{aligned}
$$

Grünwald [1867] e Letnikov [1868]

$$ {}_{\mathrm{GL}}D_{0,t}^{\alpha}u(t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t^{\alpha}} \sum_{k=0}^{N} \omega_k^{\alpha} u(t - k\Delta t), $$

$$ \omega_k^{\alpha} = (-1)^k \binom{\alpha}{k} \qquad N\Delta t = t. $$

$$ D_{0,t}^{\alpha}u(t) = \sum_{k=0}^{m-1} \frac{u^{(k)}(0)\, t^{-a+k}}{\Gamma(-\alpha + k + 1)} + \frac{1}{m-\alpha} \cdot \int_0^t (t-\tau)^{m-\alpha-1} u^{(m)}(\tau)\, d\tau, $$

$$ m - 1 \leq \alpha < m \in \mathbb{Z}^{+}. $$

Prelúdio para a definição de Riemann-Liouville

Seja $f(x)$ uma função contínua na linha real, podemos então definir a integral:

$$\begin{aligned} D^{-1}f(x) &= \int_a^x f(t)dt \\ D^{-2}f(x) &= \int_a^x \int_0^{t_2} f(t_1)dt_1 dt_2 = \int_a^x f(t_1) \int_{t_1}^x dt_2 dt_1 \\ &= \int_a^x f(t_1)(x - t_1)dt_1 \,. \end{aligned} \tag{15}$$

Então mais genericamente

$$
\begin{aligned}
D^{-3}f(x) &= \frac{1}{2}\int_a^x f(t)(x-t)^2 dt \\
D^{-4}f(x) &= \frac{1}{2\cdot 3}\int_a^x f(t)(x-t)^3 dt \\
D^{-5}f(x) &= \frac{1}{2\cdot 3\cdot 4}\int_a^x f(t)(x-t)^4 dt \\
\vdots &= \vdots \\
D^{-n}f(x) &= \frac{1}{(n-1)!}\int_a^x f(t)(x-t)^{n-1} dt \\
{}_aI_x{}^n f(x) = {}_aD_x{}^{-n} f(x) &= \frac{1}{\Gamma(n)}\int_0^x f(t)(x-t)^{n-1} dt
\end{aligned}
\quad (16)
$$

Origem da Definição de Riemann-Liouville

1869

1892 G.F.B.Riemann"Versuch Einer Allgeimenein Auffasung Der Integration und Differentiation Gesammelte Werke pp 353-366"

$$
\begin{aligned}
{}_cD_x{}^{-\nu} f(x) &= \frac{1}{\Gamma(\nu)} \int_c^x f(t)(x-t)^{\nu-1} dt + \Psi(x)\,, \\
{}_cD_x{}^{-\mu}\,{}_cD_x{}^{-\nu} f(x) &= {}_cD_x{}^{-\mu-\nu} \qquad (17)
\end{aligned}
$$

$$
\boxed{c \neq c'} \qquad (18)
$$

1869

N.Ya. Sonin "On differentiation with arbitrary index Moscow Matem. Sbornik 6(1), 1-38 "

$$D^n f(z) = \frac{n!}{2\pi i} \int_a^x \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta \,, \tag{19}$$

1869-1872

A.V.Letnikov "An explanation of the main concepts of the theory of differentiation of arbitrary index Moskow Matem. Sbornik 6, 413-445"

1884

H. Laurent "Sur le calcul des dérivées à indices quelconques Nouv. Annales de Mathematiques 3(3), 240-252"

Teoria generalizada de operadores

$D = d/dx\,, D^2 = d^2/dx^2 \cdots, D^\alpha = d^\alpha/dx^\alpha \cdots$

Riemann-Liouville [1832]

$$ {}_cD_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_c^x f(t)(x-t)^{\nu-1} dt \,, \tag{20} $$

$c = -\infty \rightarrow$ definição de Liouville
$c = 0 \rightarrow$ definição de Riemann (mais comum)

Semi-grupo

$$\boxed{{}_cI_x{}^{\mu}\,{}_cI_x{}^{\nu} \;=\; {}_cI_x{}^{\mu+\nu}}$$

$$\boxed{{}_cD_x{}^{-\mu}\,{}_cD_x{}^{-\nu}f(x) = {}_cD_x{}^{-\mu-\nu}} \tag{21}$$

Integral de RL com $f(x) = C$

$$\begin{aligned} {}_0D_x{}^{-1/2}[C] &= \frac{1}{\Gamma(1/2)} \int_0^x C(x-t)^{-1/2} dt \\ {}_0D_x{}^{-1/2}[C] &= \frac{C}{\sqrt{\pi}} \int_0^x (x-t)^{-1/2} dt \\ {}_0D_x{}^{-1/2}[C] &= \frac{2C\sqrt{x}}{\sqrt{\pi}} \end{aligned} \tag{22}$$

Integral de RL com $f(x) = \sqrt{x}$

$$ {}_0I_x^n\left[f(x)\right] = \frac{1}{\Gamma(n)}\int_0^x \left(f(y)\right)(x-y)^{n-1}\,\mathrm{d}y \qquad \boxed{n = \tfrac{1}{2}} $$

$$ {}_0I_x^{1/2}\left[\sqrt{x}\right] = {}_0D_x^{-1/2}\left[\sqrt{x}\right] = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x \sqrt{y}\,(x-y)^{\left(\frac{1}{2}\right)-1}\,\mathrm{d}y $$

$$ = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x \frac{y}{\sqrt{xy-y^2}}\mathrm{d}y $$

$$ = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x \frac{y}{\sqrt{\frac{x^2}{4}-y^2-\frac{x^2}{4}+2x\left(\frac{y}{2}\right)}}\mathrm{d}y $$

$$ = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x \frac{y}{\sqrt{\frac{x^2}{4}-\left(y-\frac{x}{2}\right)^2}}\mathrm{d}y $$

$$ y = \tfrac{1}{2}\left(x + x\sin\theta\right) \qquad \mathrm{d}y = \left(\tfrac{x}{2}\right)(\cos\theta)\mathrm{d}\theta \qquad \boxed{\begin{array}{ll} y = 0 & \theta = -\frac{\pi}{2} \\ y = x & \theta = \frac{\pi}{2}. \end{array}} $$

$\Rightarrow$

$$ {}_0D_x^{-1/2}\left[\sqrt{x}\right] = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{-\pi/2}^{\pi/2} \frac{\left(\frac{1}{2}\right)}{\sqrt{\frac{x^2}{4}-\frac{x^2}{4}\sin^2\theta}}\left(\frac{x}{2}\right)\left(\cos\theta\right)\mathrm{d}\theta $$

$$ = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_{-\pi/2}^{\pi/2} \left(\frac{1}{2}\right)\left(x + x\sin\theta\right)\mathrm{d}\theta $$

$$ = \frac{1}{2\left(\Gamma\left(\frac{1}{2}\right)\right)}\left[x\theta - x\cos\theta\right]_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} $$

$$ = \frac{\pi x}{2\left(\Gamma\left(\frac{1}{2}\right)\right)} $$

$$ = \frac{\sqrt{\pi}}{2}x $$

Integral de RL com $f(x) = x^\mu$

$$
\begin{aligned}
{}_0D_x{}^{-\nu}[x^\mu] &= \frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1} t^\mu dt \\
{}_0D_x{}^{-\nu}[x^\mu] &= \frac{1}{\Gamma(\nu)}\int_0^x \left(1-\frac{t}{x}\right)^{\nu-1} t^\mu x^{\nu-1} dt \\
{}_0D_x{}^{-\nu}[x^\mu] &= \frac{1}{\Gamma(\nu)}\int_0^x (1-u)^{\nu-1} (xu)^\mu x^{\nu-1} x du \,, \boxed{t = ux} \\
{}_0D_x{}^{-\nu}[x^\mu] &= \frac{1}{\Gamma(\nu)} x^{\mu+\nu} \int_0^1 (1-u)^{\nu-1} u^\mu du \\
{}_0D_x{}^{-\nu}[x^\mu] &= \frac{1}{\Gamma(\nu)} x^{\mu+\nu} B(\mu+1,\nu) \\
{}_0D_x{}^{-\nu}[x^\mu] &= \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} x^{\mu+\nu}
\end{aligned}
$$

(23)

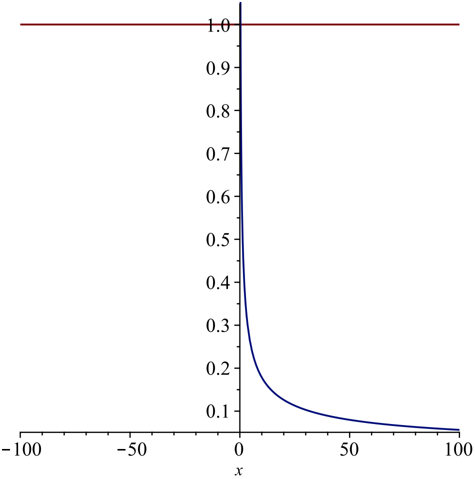
Derivada Riemann-Liouville

$$
\begin{aligned}
{}_aD_x{}^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \left(\frac{d}{dx}\right) \int_a^x f(t)(x-t)^{\alpha-1} dt \,, \\
{}_xD_b{}^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \left(-\frac{d}{dx}\right) \int_x^b f(t)(t-x)^{\alpha-1} dt \,.
\end{aligned}
\qquad (24)
$$

Derivada de RL com $f(t) = C$

$$
\begin{aligned}
{}_0D_x{}^{1/2}[C] &= \frac{1}{\Gamma(1/2)}\left(\frac{d}{dx}\right)\int_0^x C(x-t)^{-1/2}dt \\
{}_0D_x{}^{1/2}[C] &= \frac{C}{\sqrt{\pi}}\left(\frac{d}{dx}\right)\int_0^x (x-t)^{-1/2}dt \\
{}_0D_x{}^{1/2}[C] &= \frac{C}{\sqrt{\pi}}\left(\frac{d}{dx}\right)[2\sqrt{x}] \\
{}_0D_x{}^{1/2}[C] &= \frac{C}{\sqrt{\pi\, x}}
\end{aligned}
\tag{25}
$$

$C = 1$



Definição de Caputo [1967]

$$
\begin{aligned}
{}_aD_x{}^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x f'(t)(x-t)^{\alpha-1} dt \,, \\
{}_xD_b{}^{\alpha} f(x) &= -\frac{1}{\Gamma(\alpha)} \int_x^b f'(u)(t-x)^{\alpha-1} dt \,.
\end{aligned} \tag{26}
$$

Efeito Memória

1974

Qual o significado físico?

Interpretações

Talvez a mais útil seja o conceito de núcleo de memória ou efeito memória. Quando a resposta (saída) para cada tempo t depende somente do tempo inicial (entrada), dizemos que tais sistemas são sem memória.

Porém

Quando a resposta depende dos tempos prévios, dizemos que o sistema tem memória.

As derivadas usuais possuem natureza local, enquanto as derivadas fracionais possuem natureza não local.

$$
\begin{aligned}
{}_0I_t^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t-\tau)^{\alpha-1} d\tau \\
{}_0I_t^{\alpha} f(t) &= \int_0^t f(\tau) dg_t(\tau) \qquad (27)
\end{aligned}
$$

onde:

$$
\boxed{g_t(\tau) = \frac{1}{\Gamma(\alpha+1)}\{t^{\alpha} - (t-\tau)^{\alpha}\}}
$$

Sombras na Parede

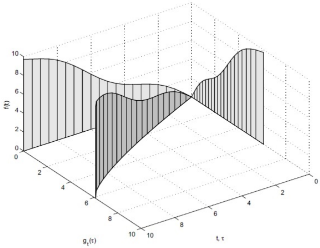


Figure 1: The "fence" and its shadows: ${}_0I_t^1 f(t)$ and ${}_0I_t^\alpha f(t)$, for $\alpha = 0.75$, $f(t) = t + 0.5\sin(t)$, $0 \le t \le 10$.

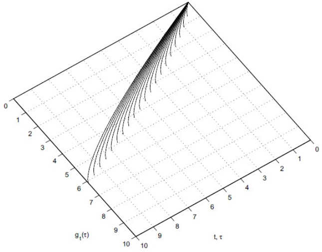


Figure 2: The process of change of the fence basis shape for ${}_0I_t^\alpha f(t)$, $\alpha = 0.75$, $0 \le t \le 10$.

Sombras na Parede

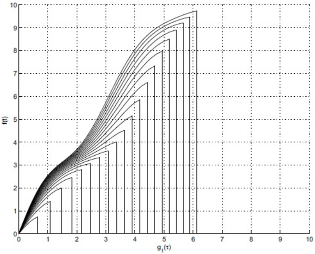


Figure 3: Snapshots of the changing "shadow" of changing "fence" for ${}_0I_t^{\alpha} f(t)$, $\alpha = 0.75$, $f(t) = t + 0.5\sin(t)$, $0 \leq t \leq 10$, with the time interval $\Delta t = 0.5$ between the snapshops.

Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 5, NUMBER 4 (2002) ISSN 1311-0454

GEOMETRIC AND PHYSICAL INTERPRETATIO
OF FRACTIONAL INTEGRATION AND
FRACTIONAL DIFFERENTIATION *

Igor Podlubny

Métodos Variacionais

$$
\begin{aligned}
S &= \frac{1}{\Gamma(\alpha)} \int_a^b L(t, {}_aD_t{}^{\beta} q, {}_tD_b{}^{\gamma} q)(t-\tau)^{\alpha-1} d\tau \\
\delta S &= 0 \\
\frac{\partial L}{\partial q} &+ \frac{1}{(t-\tau)^{\alpha-1}} \left[{}_tD_b{}^{\beta} \left(\frac{\partial L}{\partial ({}_aD_t{}^{\beta} q)} (t-\tau)^{\alpha-1} \right) \right] + \\
&+ \frac{1}{(t-\tau)^{\alpha-1}} \left[{}_aD_t{}^{\beta} \left(\frac{\partial L}{\partial ({}_tD_b{}^{\beta} q)} (t-\tau)^{\alpha-1} \right) \right] = 0
\end{aligned}
\tag{28}
$$

$\beta = \gamma = 1$

$$\begin{aligned}\frac{\partial L}{\partial q} &- \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \left(\frac{\alpha - 1}{t - \tau}\right)\frac{\partial L}{\partial \dot{q}} = 0 \\ \frac{\partial L}{\partial q} &- \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial \mathcal{F}}{\partial \dot{q}} = 0 \qquad (29)\end{aligned}$$

Aplicações

1)Hereditariedade de materiais: Viscoelasticidade
2)Difusão em meios porosos
3)Meios Fractais

AN INTRODUCTION TO THE FRACTIONAL CALCULUS AND FRACTIONAL DIFFERENTIAL EQUATIONS

KENNETH S. MILLER
Mathematical Consultant
Formerly Professor of Mathematics
New York University

BERTRAM ROSS
University of New Haven

APPLICATIONS

OF

FRACTIONAL CALCULUS IN PHYSICS

Editor

R. Hilfer

Universität Mainz and Universität Stuttgart
Germany

Outras Definições

Liouville derivative:

$$\mathrm{D}^{\alpha}\left[f(x)\right] = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{x}(x-\xi)^{-\alpha}f(\xi)\,\mathrm{d}\xi, \qquad -\infty < x < +\infty.$$

Liouville left-sided derivative:

$$\mathrm{D}^{\alpha}_{0^+}\left[f(x)\right] = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_{0}^{x}(x-\xi)^{-\alpha+n-1}f(\xi)\,\mathrm{d}\xi, \qquad x > 0.$$

Liouville right-sided derivative:

$$\mathrm{D}^{\alpha}_{-}\left[f(x)\right] = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_{x}^{\infty}(x-\xi)^{-\alpha+n-1}f(\xi)\,\mathrm{d}\xi, \qquad x < \infty.$$

Riemann-Liouville left-sided derivative:

$${}^{\mathrm{RL}}\mathrm{D}^{\alpha}_{a^+}\left[f(x)\right] = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_{a}^{x}(x-\xi)^{n-\alpha-1}f(\xi)\,\mathrm{d}\xi, \qquad x \geq a.$$

Riemann-Liouville right-sided derivative:

$${}^{\mathrm{RL}}\mathrm{D}^{\alpha}_{b^-}\left[f(x)\right] = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\int_{x}^{b}(\xi-x)^{n-\alpha-1}f(\xi)\,\mathrm{d}\xi, \qquad x \leq b.$$

Caputo left-sided derivative:

$${}_{*}\mathrm{D}^{\alpha}_{a^+}\left[f(x)\right] = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}(x-\xi)^{n-\alpha-1}\frac{\mathrm{d}^n}{\mathrm{d}\xi^n}\left[f(\xi)\right]\mathrm{d}\xi, \qquad x \geq a.$$

Caputo right-sided derivative:

$${}_{*}\mathrm{D}^{\alpha}_{b^-}\left[f(x)\right] = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_{x}^{b}(\xi-x)^{n-\alpha-1}\frac{\mathrm{d}^n}{\mathrm{d}\xi^n}\left[f(\xi)\right]\mathrm{d}\xi, \qquad x \leq b.$$

Riemann-Liouville left-sided integral:

$${}^{\mathrm{RL}}\mathrm{I}^{\alpha}_{a^+}\left[f(x)\right] = \frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-\xi)^{\alpha-1}f(\xi)\,\mathrm{d}\xi, \qquad x \geq a.$$

Riemann-Liouville right-sided integral:

$${}^{\mathrm{RL}}\mathrm{I}^{\alpha}_{b^-}\left[f(x)\right] = \frac{1}{\Gamma(\alpha)}\int_{x}^{b}(\xi-x)^{\alpha-1}f(\xi)\,\mathrm{d}\xi, \qquad x \leq b.$$

Grünwald-Letnikov left-sided derivative:

$$ {}^{GL}D^{\alpha}_{a^+}[f(x)] = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\lfloor n\rfloor}(-1)^k\frac{\Gamma(\alpha+1)f(x-kh)}{\Gamma(k+1)\Gamma(\alpha-k+1)}, \quad nh = x-a. $$

Grünwald-Letnikov right-sided derivative:

$$ {}^{GL}D^{\alpha}_{b^-}[f(x)] = \lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\lfloor n\rfloor}(-1)^k\frac{\Gamma(\alpha+1)f(x+kh)}{\Gamma(k+1)\Gamma(\alpha-k+1)}, \quad nh = b-x. $$

Weyl derivative:

$$ {}_xD^{\alpha}_{\infty}[f(x)] = D^{\alpha}_{-}[f(x)] = (-1)^m\left(\frac{d}{d\xi}\right)^n\left[{}_xW^{\alpha}_{\infty}[f(x)]\right]. $$

Marchaud derivative:

$$ D^{\alpha}_{+}[f(x)] = \frac{\alpha}{\Gamma(1-\alpha)}\int_{-\infty}^{x}\frac{f(x)-f(\xi)}{(x-\xi)^{1+\alpha}}d\xi. $$

Marchaud left-sided derivative:

$$ D^{\alpha}_{+}[f(x)] = \frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{\infty}\frac{f(x)-f(x-\xi)}{\xi^{1+\alpha}}d\xi. $$

Marchaud right-sided derivative:

$$ D^{\alpha}_{-}[f(x)] = \frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{\infty}\frac{f(x)-f(x+\xi)}{\xi^{1+\alpha}}d\xi. $$

Hadamard derivative [42]:

$$ D^{\alpha}_{+}[f(x)] = \frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{x}\frac{f(x)-f(\xi)}{[\ln(x/\xi)]^{1+\alpha}}\frac{d\xi}{\xi}. $$

Chen left-sided derivative:

$$ D^{\alpha}_{c}[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{c}^{x}(x-\xi)^{-\alpha}f(\xi)\,d\xi, \quad x>c. $$

Chen right-sided derivative:

$$ D^{\alpha}_{c}[f(x)] = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x}^{c}(\xi-x)^{-\alpha}f(\xi)\,d\xi, \quad x<c. $$

Davidson-Essex derivative [15]:

$$ D^{\alpha}_{0}[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d^{n+1-k}}{dx^{n+1-k}} \times\int_{0}^{x}(x-\xi)^{-\alpha}\frac{d^k}{d\xi^k}[f(\xi)]\,d\xi. $$

Coimbra derivative [43–45]:

$$ D^{\alpha(x)}_{0}[f(x)] = \frac{1}{\Gamma(1-\alpha(x))} \times\left\{\int_{0}^{x}(x-\xi)^{-\alpha(x)}\frac{d}{d\xi}[f(\xi)]\,d\xi + f(0)\,x^{-\alpha(x)}\right\}. $$

Canavati derivative:

$$ {}_aD^{\nu}_{x}[f(x)] = \frac{1}{\Gamma(1-\mu)}\frac{d}{dx}\int_{0}^{x}(x-\xi)^{\mu}\frac{d^n}{d\xi^n}[f(\xi)]\,d\xi, \quad n=\lfloor\nu\rfloor, \quad \mu=n-\nu. $$

Jumarie derivative, $n=1$:

$$ D^{\alpha}_{x}[f(x)] = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n} \times\int_{0}^{x}(x-\xi)^{n-\alpha-1}[f(\xi)-f(0)]\,d\xi. $$

Riesz derivative:

$$ D^{\alpha}_{x}[f(x)] = -\frac{1}{2\cos(\alpha\pi/2)}\frac{1}{\Gamma(\alpha)}\frac{d^n}{dx^n} \cdot\left\{\int_{-\infty}^{x}(x-\xi)^{n-\alpha-1}f(\xi)\,d\xi + \int_{x}^{\infty}(\xi-x)^{n-\alpha-1}f(\xi)\,d\xi\right\}. $$

Cossar derivative:

$$ D^{\alpha}_{-}[f(x)] = -\frac{1}{\Gamma(1-\alpha)}\lim_{N\to\infty}\frac{d}{dx}\int_{x}^{N}(\xi-x)^{-\alpha}f(\xi)\,d\xi. $$

Local fractional Yang derivative [40]:

$$ D^{\alpha}_{-}[f(x)]\big|_{x=x_0} = \lim_{x\to x_0}\frac{\Delta^{\alpha}[f(x)-f(x_0)]}{(x-x_0)^{\alpha}}. $$

Left Riemann-Liouville derivative of variable fractional order:

$$ {}_aD^{\alpha(\cdot,\cdot)}_{x}[f(x)] = \frac{d}{dx}\int_{a}^{x}(x-\xi)^{-\alpha(\xi,x)}f(\xi)\frac{d\xi}{\Gamma[1-\alpha(\xi,x)]}. $$

Right Riemann-Liouville derivative of variable fractional order:

$$ {}_xD^{\alpha(\cdot,\cdot)}_{b}[f(x)] = \frac{d}{dx}\int_{x}^{b}(\xi-x)^{-\alpha(\xi,x)}f(\xi)\frac{d\xi}{\Gamma[1-\alpha(\xi,x)]}. $$

Left Caputo derivative of variable fractional order:

$$ {}_aD^{\alpha(\cdot,\cdot)}_{x}[f(x)] = \int_{a}^{x}(x-\xi)^{-\alpha(\xi,x)}\frac{d}{d\xi}f(\xi)\frac{d\xi}{\Gamma[1-\alpha(\xi,x)]}. $$

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2014, Article ID 238459, 6 pages
http://dx.doi.org/10.1155/2014/238459

Review Article

A Review of Definitions for Fractional Derivatives and Integral

Edmundo Capelas de Oliveira[1] and José António Tenreiro Machado[2]

[1] *Department of Applied Mathematics, IMECC-UNICAMP, 13083-859 Campinas, SP, Brazil*
[2] *Institute of Engineering, Polytechnic of Porto, Department of Electrical Engineering, Rua Dr. Antonio Bernardino de Almeida 431, 4200-072 Porto, Portugal*

Correspondence should be addressed to José António Tenreiro Machado; jtenreiromachado@gmail.com

Received 30 January 2014; Accepted 18 May 2014; Published 10 June 2014

Academic Editor: Riccardo Caponetto

This paper presents a review of definitions of fractional order derivatives and integrals that appear in mathematics, physics, and engineering.

1. Introduction

In 1695, l'Hôpital sent a letter to Leibniz. In his message, an important question about the order of the derivative emerged: What might be a derivative of order 1/2? In a

after his death. We also note that both Liouville and Riemann formulations carry with them the so-called complementary function, a problem to be solved. Grünwald [7] and Letnikov [8], independently, developed an approach to noninteger order derivatives in terms of a convenient convergent series