# Group Representations in Quantum Mechanics and Group Theory

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## Sumário







## Group Representations - Preliminary Definitions and Properties

**Definition**

A group representation is a group homomorphism from an original group $G$ to a space of linear transformations $L(V)$ defined in a vector space $V$.

$$\Gamma : G \longrightarrow L(V)$$

**Example**

$$V = \mathbb{C}^n \,, \quad L(V) = GL(n, \mathbb{C})$$

**Example**

Symmetric group in three letters

$$G = S_3 = \{(\,), (1,2), (2,3), (1,3), (1,2,3), (2,3,1)\}$$





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Symmetric group in three letters

$$G = S_3 = \{(\,), (1,2), (2,3), (1,3), (1,2,3), (2,3,1)\}$$

$$(\,) \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(1,2,3) \longmapsto \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$$

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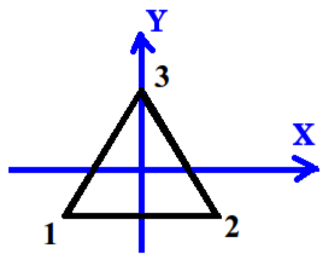
$$(1,2) \longmapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(2,3) \longmapsto \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$$

$$(1,3) \longmapsto \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$



We may classify group representations according to

- faithfulness $\rightarrow$ If the representation is an isomorphism, it is called faithfull.
- reducibility $\rightarrow$ If the representation can be reduced to block-diagonal form, it is called reducible.

### Definition

For a group representation $R$ of a group $G$, we have:

i) When $R$ is a one-dimensional representation, it is *irreducible*.

ii) In case the dimension of $R$ is greater than one, if there exists a similarity transformation $R_B$ in which all the matrices are written in block-diagonal form, the representation $R$ is called *reducible*, otherwise $R$ is *irreducible*.

Bloco-diagonal form: $\begin{pmatrix} \times & \times & \times & . & . & . \\ \times & \times & \times & . & . & . \\ \times & \times & \times & . & . & . \\ . & . & . & \times & . & . \\ . & . & . & . & \times & \times \\ . & . & . & . & \times & \times \end{pmatrix}$





### Definition

Let $G$ be a group and $R : G \longrightarrow GL(n, \mathbb{C})$ a representation of $G$. Given the matrix $B \in GL(n, \mathbb{C})$, the application

$$\begin{aligned} R_B : G &\longrightarrow GL(n, \mathbb{C}) \, , \\ g &\mapsto BR(g)B^{-1} \, , \end{aligned}$$

is called *similarity transformation* of $R$ generated by $B$.

A similarity transformation for a group representation is, by its turn, also a group representation of the same group $G$.

### Definition

Two representations $R_1$ and $R_2$ of a group $G$ are called *similar* ou *equivalent* when there exists a matrix $B$ such that $R_2$ is a similarity transformation of $R_1$ generated by $B$.





A similarity transformation is always invertible. Indeed

$$R_B(g) = BR(g)B^{-1} \iff B^{-1}R_B(g)B = R(g) \, .$$

Similarity transformatrions appear naturaly in base changes. Assume $a$ and $b$ are two vectors related by $b = Aa$ and perform a chnage of basis $a \mapsto a' = Ba$ e $b \mapsto b' = Bb$. Then

$$b = Aa \rightarrow Bb = BAa \rightarrow Bb = BAB^{-1}Ba \rightarrow b' = A'a'$$

with

$$A' = BAB^{-1} \, .$$

That means, under the change of basis defined by $B$ the matrix $A$ tranforms as $A \mapsto A' = BAB^{-1}$.
An important property which remains invariant under similarity transformations is the matricial trace

$$\mathrm{tr}(BAB^{-1}) = \mathrm{tr}(B^{-1}BA) = \mathrm{tr}\, A \, .$$





### Example

Let $G$ be the cyclic group $C_4 = \{e, a, a^2, a^3\} \cong \mathbb{Z}_4$, of order four, consider now the application

$$e \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad a \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ,$$

$$a^2 \longmapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad a^3 \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

This is an isomorphism between $C_4$ and a multiplicative matricial group. This is a faithful matricial representation of $C_4$; the above matrices reproduce the group multiplicative table.

In the example of the $C_4$ representation, at first we cannot tell its irreducibility, because the matrices associated to the elements $a$ and $a^3$ are not diagonal.

$$e \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$a^2 \longmapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad a^3 \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

However, a similarity transformation with

$$B = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

diagonalizes immediately all matrices

$$e \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a \longrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$
$$a^2 \longrightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad a^3 \longrightarrow \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

and we see the representa is reducible. Indeed, irreducible representations of Abelian groups must be one dimensional.





## Character Tables

Given a group $G$ and a linear representation $\Gamma$, we know that the trace of the group element associated matrices are invariant by similarity transformations. We may go further and observe that this trace is the same for all elements in a conjugation class. Therefore we may define the character for group elements as well as for class congugation.

That is:
in practice the character is the **matricial trace.**





## Character Tables

**Definition**

Given a representation $k$ of a finite group $G$, we define its character $\chi^k : G \to \mathbb{C}$ as the complex function

$$\chi^k_\alpha = \operatorname{tr} A^k_\alpha \, .$$





Given a representation of a finite group $G$, the elements of a conjugation class possess all the same character.

The character function of a finite group satisfies

$$\chi^k(g_\alpha^{-1}) = [\chi^k(g_\alpha)]^*$$

for all $g_\alpha \in G$.

**Definition**

Given a representation $k$ of a group $G$, the character associated to a conjugation class $\mathcal{C}_{\bar\alpha}$ is the character of an element $g_\alpha \in \mathcal{C}_{\bar\alpha}$.

### Example

The Quaternium Group (Group of Quaternium Units)

| $k$ | $1$ | $-1$ | $\{\pm i\}$ | $\{\pm j\}$ | $\{\pm k\}$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | −1 | −1 |
| 3 | 1 | 1 | −1 | 1 | −1 |
| 4 | 1 | 1 | −1 | −1 | 1 |
| 5 | 2 | −2 | 0 | 0 | 0 |

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i \mapsto \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad j \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad k \mapsto \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$
$$-1 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad -i \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad -j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad -k \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$





### Example

Symmetric Group in 4 Letters

| $k$ | 1 | 6(12) | 8(123) | 6(1234) | 3(12)(34) |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | −1 | 1 | −1 | 1 |
| 3 | 2 | 0 | −1 | 0 | 2 |
| 4 | 3 | 1 | 0 | −1 | −1 |
| 5 | 3 | −1 | 0 | 1 | −1 |





### Example

The Alternating Group in four letters

| $k$ | 1 | 3(12)(34) | 4(123) | 4(132) |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | $\omega$ | $\omega^2$ |
| 3 | 1 | 1 | $\omega^2$ | $\omega$ |
| 4 | 3 | −1 | 0 | 0 |

with $\omega = e^{2\pi i/3}$





## The Group of the Hamiltonian

Two operators $A$ and $B$ are considered similar when related bya a similarity transformation

$$B = RAR^{-1}$$

for some $R$.

The Hamiltonian $\mathcal{H}$ is invariant under $R$ if

$$\mathcal{H} = R\mathcal{H}R^{-1}$$

which is equivalent to

$$[\mathcal{H}, R] = 0$$

The Hamiltonian $\mathcal{H}$ is invariant under $R$ if

$$\mathcal{H} = R\mathcal{H}R^{-1}$$

which is equivalent to

$$[\mathcal{H}, R] = 0$$

We call $R$ a Hamiltonian symmetry. We gather all Hamiltonian symmetries in the set $G(\mathcal{H})$ which happens to be a group

$$G(\mathcal{H}) = \{R \in L(\mathcal{V}); R\mathcal{H} = \mathcal{H}R\}$$

We call $G(\mathcal{H})$ the Hamiltonian Symmetry Group.





$G(\mathcal{H})$ is a group:

Indeed, if $R_\alpha$ are $R_\beta$ Hamiltonian symmetries, we have

$$\begin{aligned}(R_\alpha R_\beta)\mathcal{H} &= R_\alpha(R_\beta\mathcal{H}) \\ &= R_\alpha(\mathcal{H}R_\beta) \\ &= (R_\alpha\mathcal{H})R_\beta \\ &= (\mathcal{H}R_\alpha)R_\beta = \mathcal{H}(R_\alpha R_\beta)\,,\end{aligned}$$

and the product $R_\alpha R_\beta$ is also a symmetry of the Hamiltonian.

The identity operator $I$ is a symmetry of the Hamiltonian, as $I$ commutes with any operator.





In summary, we have the following ingredients

$$\begin{gathered}\text{A Hilbert Space} \longrightarrow \mathcal{V} \\ \text{An Hermitian Operator} \longrightarrow \mathcal{H} \in L(\mathcal{V}) \\ \text{The Group of the Hamiltonian} \longrightarrow G(\mathcal{H}) \\ \text{Symmetry Operators} \longrightarrow R \in G(\mathcal{H})\end{gathered}$$

Consider the eigenvalue equation for $\mathcal{H}$ in the Hilbert Space $\mathcal{V}$

$$\mathcal{H}\varphi = E\varphi\,,\ \ \varphi \in \mathcal{V}$$

For a given eigenvalue $E$, we define the subspace

$$V(E) = \{\varphi \in \mathcal{V};\ \mathcal{H}\varphi = E\varphi\}$$





For a given $\varphi \in V(E)$, we define $\mathcal{E}_\varphi$ as the subspace generated by all $R\varphi$ with $R \in G(\mathcal{H})$.
We assume normal degeneracy, which means

$$\mathcal{E}_\varphi = V(E)\,, \forall\varphi \in V(E)$$

Let $l = dimV(E)$. The eigenvalue $E$ defines an $l$-dimensional representation of $G(\mathcal{H})$.
Explicitly, choose a basis $\{\varphi_1, \ldots, \varphi_l\}$ for $V(E$ and define the row vector

$$[R\varphi] = (R\varphi_1 \quad R\varphi_2 \quad \ldots \quad R\varphi_l)$$

as each $R\varphi_i$ is a linear combination from $\varphi_i$, we have

$$[R\varphi] = [\varphi]\Gamma(R)$$

That is, we have a $l$-dimensional representation of the group $G(\mathcal{H})$.

Note that the relation

$$[R\varphi] = [\varphi]\Gamma(R)$$

may be rewritten in components as

$$R\varphi_i = \sum_{j=1}^{l} \varphi_j (\Gamma(R))_{ji}$$

As the transformation $R$ must preserve the norm

$$||R\varphi|| = ||\varphi||$$

$\Gamma(R)$ is a unitary representation of the group $G(\mathcal{H})$.





## Bloch's Theorem

Consider a onedimensional crystal with $N$ repetition unities (sites)

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\varphi = E\varphi$$

Periodic Potential

$$V(x+a) = V(x)$$

$$\varphi(x+Na) = \varphi(x)$$

Hamiltonian Operator

$$\mathcal{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

The vectors in the Hilbert Space are complex functions $\psi : \mathbb{R} \to \mathbb{C}$, written simply as $\psi(x)$ (the wave function)





Define the translation operator

$$R_a\psi(x) = \psi(x+a)$$

Since $d(x+a) = dx$ e $V(x+a) = V(x)$, we have the symmetry

$$R_a\mathcal{H} = \mathcal{H}R_a$$





Since $R_a$ is a Hamiltonian symmetry, so is $R_a^n$.
Thus $R_a^N = E$ and we have the symmetry group

$$G = \{R_a, R_a^2, \ldots, R_a^{N-1}, R_a^N\}$$

as a subgroup of $G(\mathcal{H})$. Since $G$ is Abelian, the irreducible representations are one dimensional. The characters must obey $[\chi(R_a)]^N = 1$, thus

$$\begin{gathered}\chi^{(1)}(R_a) = 1 \\ \chi^{(2)}(R_a) = e^{2\pi i/N} \\ \chi^{(3)}(R_a) = e^{2.(2\pi i/N)} \\ \vdots \\ \chi^{N}(R_a) = e^{(N-1)}.(2\pi i/N)\end{gathered}$$

The character table maybe written as

| | $\{E\}$ | $\{R_a\}$ | $\{R_a^2\}$ | $\cdots$ | $\{R_a^{N-2}\}$ | $\{R_a^{N-1}\}$ |
|---|---|---|---|---|---|---|
| $\Gamma_1$ | 1 | 1 | 1 | $\cdots$ | 1 | 1 |
| $\Gamma_2$ | 1 | $\omega$ | $\omega^2$ | $\cdots$ | $\omega^{N-2}$ | $\omega^{N-1}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ | $\vdots$ |
| $\Gamma_{N-1}$ | 1 | $\omega^{N-2}$ | $\omega^{N-4}$ | $\cdots$ | $\omega^4$ | $\omega^2$ |
| $\Gamma_N$ | 1 | $\omega^{N-1}$ | $\omega^{N-2}$ | $\cdots$ | $\omega^2$ | $\omega$ |

With a simple calculation, we may prove one of the two results of Bloch's Theorem, namely the one with asserts that the solutions are periodic wave functions enveloped by plane waves

$$\varphi_n(x) = e^{ik_n x} u_n(x)\,,\ \text{com}\ u_n(x) = u_n(x+a)$$





## Supersymmetry in Quantum Mechanics

Consider a spinless mass $m$ particle on a line subjected to a one dimensional real potential $V(x)$. Its quantum mechanical description amounts to constructing an infinite dimensional Hilbert space $\mathcal{E}$ of kets $|\psi>$ representing the possible particle states. The particle dynamical evolution is governed by the Hamiltonian

$$\mathbf{H} = \frac{\mathbf{P}^2}{2m} + V(\mathbf{X})\,, \tag{1}$$

where $\mathbf{P}$ and $\mathbf{X}$ are, correspondingly, the momentum and position Hermitian operators acting on $\mathcal{E}$. Since the potential is time independent, the well-known separation of variables technique can be applied, leading, in the position basis, to the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x)\,. \tag{2}$$





$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x)\,. \tag{3}$$

This is an ordinary second order differential equation for the complex wave function $\psi(x)$ corresponding to the ket $|\psi> \in \mathcal{E}$. We may also interpret (3) as an eigenvalue-eigenvector problem. Given a potential $V(x)$, we seek for complex eigenfunctions $\psi(x)$ and corresponding real eigenvalues $E$. The real numbers $E$, being eigenvalues of the Hamiltonian, represent the energy spectrum of the theory.

Let $\psi_0(x)$ be the ground state solution of (3), corresponding to the minimal energy $E_0$. Redefining the potential as $V_-(x) \equiv V(x) - E_0$ we may write

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_0(x) + V_-(x)\psi_0(x) = 0 \tag{4}$$

for the ground state and all energy levels get downshifted by $E_0$.





Associated to the potential $V_-$ we can define the corresponding Hamiltonian $H^-$ given by

$$H^- \equiv \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V_-\,. \tag{5}$$

Labeling eigenfunctions and eigenvalues by the subscript $n$ we have explicitly

$$H^-\psi_n^-(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_n^-(x) + V_-(x)\psi_n^-(x) = E_n^-\psi_n^-(x)\,, \tag{6}$$

with $E_n^- \equiv E_n - E_0$. As defined above the ground state of $H^-$ can be readily checked to have zero energy

$$H^-\psi_0^- = 0\,. \tag{7}$$

Naturally the eigenstates of (3) are the same as those of (6) and particularly $\psi_0 = \psi_0^-$.

Aiming to obtain a supersymmetric partner for the potential $V_-$, we shall now introduce the elements of supersymmetry in the theory. Inspired by the well-known creation/anihilation operator technique of the harmonic oscillator we begin factorizing the second order operator $H^-$ into

$$H^- = A^+ A^- , \tag{8}$$

with,

$$\begin{aligned} A^- &\equiv \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)\,, \\ A^+ &\equiv \frac{-\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)\,, \end{aligned} \tag{9}$$

where $W(x)$ is a solution of the Riccati non-linear first order differential equation

$$V_- = W^2(x) - \frac{\hbar}{\sqrt{2m}} W'(x)\,. \tag{10}$$





The quantity $W(x)$ is called the superpotential associated to the original potential $V(x)$ in (3) and satisfies the commutation relation

$$\left[A^-, A^+\right] = \frac{2\hbar}{\sqrt{2m}} W'(x)\,. \tag{11}$$

Notice that if a ground state eigenfunction $\psi_0$ satisfying (4) for a particular one-dimensional potential $V(x)$ is known, one can immediately write

$$W(x) = -\frac{\hbar}{\sqrt{2m}}\frac{\psi_0'(x)}{\psi_0(x)} \tag{12}$$

Switching the order between $A^-$ and $A^+$ in (9) we define the operator

$$H^+ \equiv A^- A^+ = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_+\,, \tag{13}$$

$$V_+ \equiv \frac{\hbar}{\sqrt{2m}} W' + W^2\,. \tag{14}$$





Here $H^+$ and $V_+$ are known respectively as the SUSY partners of $H^-$ and $V_-$. As can be easily checked, $A^+$ and $A^-$ are the adjoint of each other, while both Hamiltonians $H^+$ and $H^-$ are Hermitian semi-positive-definite operators. In the following, let us figure out how the eigenvalues and eigenfunctions of $H^-$ and $H^+$ are interrelated. Denoting the eigenfunctions of $H^-(H^+)$ by $\psi_n^-(\psi_n^+)$ we write

$$\begin{aligned} H^-\psi_n^-(x) &= E_n^-\psi_n^-(x)\,, \\ H^+\psi_n^+(x) &= E_n^+\psi_n^+(x)\,. \end{aligned} \tag{15}$$

Concerning solutions $\phi^\pm$ to $A^+\phi^+ = A^-\phi^- = 0$, we may write

$$\phi^\pm \sim \exp\left[\mp\frac{\sqrt{2m}}{\hbar}\int W(x)dx\right]\,, \tag{16}$$

and particularly $\phi^+ \sim (\phi^-)^{-1}$. That means if $\phi^-$ is normalizable, $\phi^+$ is not. We assume $\phi^-$ to be normalizable and consider $\phi^- = \psi_0^-$ which satisfies

$$H^-\psi_0^- = A^+(A^-\psi_0^-) = 0\,. \tag{17}$$





Explicitly we write

$$\psi_0^-(x) = C\exp\left[-\frac{\sqrt{2m}}{\hbar}\int W(x)dx\right]\,, \tag{18}$$

with

$$\int dx\,|\psi_0^-|^2 = 1\,. \tag{19}$$

Therefore, considering the eigenvalues in (15) ordered by increasing value of energies, we must have $E_0^- = 0$ and $E_0^+ > 0$. Observing that

$$H^+(A^-\psi_n^-) = E_n^-(A^-\psi_n^-)\,, \tag{20}$$

and comparing with the second equation of (15) we see that (i) the spectrum of $H^+$ coincides with that of $H^-$ with the sole exception of $E_0^- = 0$ and (ii) the eigenfunctions of $H^+$ are proportional to $A^-\psi_n^-$.

We thus write

$$E_n^+ = E_{n+1}^- \,, \ n \geq 0 \,, \tag{21}$$

and

$$\psi_n^+ = \frac{1}{\sqrt{E_{n+1}^-}} A^- \psi_{n+1}^- \,, \ n \geq 0 \,. \tag{22}$$

By applying $A^+$ to both sides of the last equation it can be inverted to

$$\psi_{n+1}^- = \frac{1}{\sqrt{E_n^+}} A^+ \psi_n^+ \,, \ n \geq 0 \,. \tag{23}$$

We see that the $A^-$ and $A^+$ operators connect $H^-$ and $H^+$ eigenstates with the same energy. Knowledge of the eigenstates and eigenvalues of one of the Hamiltionians $H^\pm$ leads to the knowledge of the corresponding solution for its partner. In the following sections we apply this formalism to specific one dimensional potentials.





## Infinite Square Well Potential

In this section we illustrate the previously discussed central SUSYQM ideas in the simple infinite square well potential, also known as "particle in a box potential". We start with the time-independent Schrödinger equation (3) with the potential $V(x)$ given by

$$V(x) = \begin{cases} 0 \,, & |x| \leq a \,; \\ \infty \,, & |x| > a \,. \end{cases} \tag{24}$$

The positive real parameter $a$, with length dimension, characterizes the well potential width. The wave function vanishes for $|x| \geq a$ confining thus the particle inside the "box" $|x| < a$. For $|x| < a$, equation (3) reduces to

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi(x) \,. \tag{25}$$





Non-positive energy eigenvalues lead to wave solutions which cannot match continuity at $|x| = a$, unless $\psi \equiv 0$ which is not an allowed eigenvector by definition. Therefore we must have $E > 0$. Defining $k = \sqrt{\frac{2mE}{\hbar^2}}$ we write the general solution for (25) as

$$\psi(x) = A \cos kx + B \sin kx \,. \tag{26}$$

The boundary condition $\psi(a) = \psi(-a) = 0$ enforces either $B = 0$ with $ka = (2n+1)\pi/2$ or $A = 0$ with $ka = n\pi$ for natural $n$.





Therefore, labelling the solutions by $n \in \mathbb{N}$ in increasing order of energy value, we have

$$\begin{aligned} E_n &= \frac{\pi^2\hbar^2}{8ma^2} n^2 \,, \quad n = 1, 2, 3, \ldots \,, \\ \psi_n &= \begin{cases} B_n \cos\left[\frac{n\pi x}{2a}\right] \,, & \text{for } n = 1, 3, 5, \ldots \,, \\ B_n \sin\left[\frac{n\pi x}{2a}\right] \,, & \text{for } n = 2, 4, 6, \ldots \,, \end{cases} \end{aligned} \tag{27}$$

By subtracting the ground state energy and shifting $n$ to $n+1$ we get

$$\begin{aligned} E_n^- &= \frac{\pi^2\hbar^2}{8ma^2} n(n+2) \,, \quad n = 0, 1, 2, \ldots \,, \\ \psi_n^- &= \begin{cases} C_n \cos\left[\frac{(n+1)\pi x}{2a}\right] \,, & \text{for } n = 0, 2, 4, \ldots \,, \\ C_n \sin\left[\frac{(n+1)\pi x}{2a}\right] \,, & \text{for } n = 1, 3, 5, \ldots \end{cases} \end{aligned} \tag{28}$$

according to the previous SUSY notation.

The superpotential can be readily obtained from (12) as

$$W(x) = \frac{\hbar\pi}{\sqrt{8ma^2}} \tan\left(\frac{\pi x}{2a}\right) , \tag{29}$$

and the SUSY partner potential (14) reads

$$V_+(x) = \frac{\hbar^2\pi}{8ma^2}\left[2\sec^2\left(\frac{\pi x}{2a}\right) - 1\right] . \tag{30}$$

The potential $V_+(x)$ in (30) can be promptly recognized as the Pöschl-Teller potential.
Now we may use our knowledge of the solution to the infinite square well potential (28) and its corresponding superpotential (29) to generate the set of solutions (23) to the Pöschl-Teller potential (30). For instance, for the first three eigenfunctions of $H^+$, an explicit calculation using (23) leads to

$$\begin{aligned} \psi_0^+ &= \cos^2\left(\frac{\pi x}{2a}\right) \\ \psi_1^+ &= \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi x}{2a}\right) \\ \psi_2^+ &= 4\cos^4\left(\frac{\pi x}{2a}\right) - 5\sin^2\left(\frac{\pi x}{a}\right) \end{aligned} \tag{31}$$





Further, the corresponding eigenvalues, obtained from (21), are easily found to be $E_0^+ = \frac{3\pi^2\hbar^2}{8ma^2}$, $E_1^+ = \frac{8}{3}E_0^+$ and $E_2^+ = 5E_0^+$.
Thus we see in this example that knowledge of the solution of the simpler eigenvalue problem for $H^-$ enables one to readily solve the more involving Pöschl-Teller potential problem.





## Supersymetric Operators

The partner Hamiltonians are given by

$$H_- = A^+A^-$$

$$H_+ = A^-A^+$$

Note that the expectation values for these operators are always non-negative

$$\begin{aligned} <\phi|H_\mp|\phi> &= <\phi|A^\pm A^\mp|\phi> \\ &= (<\phi|A^\pm)(A^\mp|\phi>) = ||A^\mp|\phi>||^2 \geq 0 \end{aligned}$$

Now introduce two new operators $Q^\pm$ given by

$$Q^- \equiv \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix} \quad \text{and} \quad Q^+ \equiv \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix}$$





$$Q^- \equiv \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix} \quad \text{and} \quad Q^+ \equiv \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix}$$

which upon multiplication result in

$$Q^-Q^+ \equiv \begin{pmatrix} 0 & 0 \\ 0 & A^-A^+ \end{pmatrix} \quad Q^+Q^- \equiv \begin{pmatrix} A^+A^- & 0 \\ 0 & 0 \end{pmatrix}$$

that is

$$Q^-Q^+ + Q^+Q^- = \begin{pmatrix} A^+A^- & 0 \\ 0 & A^-A^+ \end{pmatrix} = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix}$$

Define a Hamiltonian in matrix form as

$$H \equiv \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} = \begin{pmatrix} A^+A^- & 0 \\ 0 & A^-A^+ \end{pmatrix}$$

We promptly note that

$$H = Q^-Q^+ + Q^+Q^- = \{Q^-, Q^+\}$$

Furthermore

$$(Q^\pm)^2 = 0$$

and

$$[Q^\pm, H] \equiv Q^\pm H - HQ^\pm = 0$$

The property $[Q^\pm, H] = 0$ signals an underlying symmetry.





By defining the dublets

$$\begin{pmatrix} \psi_n^-(x) \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ \psi_n^+(x) \end{pmatrix}$$

we see that $Q^\pm$ act like ladder operators

$$\begin{aligned} Q^- \begin{pmatrix} \psi_n^-(x) \\ 0 \end{pmatrix} &\equiv \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix} \begin{pmatrix} \psi_n^-(x) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ A^-\psi_n^-(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \psi_{n-1}^+(x) \end{pmatrix} \\ Q^+ \begin{pmatrix} 0 \\ \psi_n^+(x) \end{pmatrix} &\equiv \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \psi_n^+(x) \end{pmatrix} \\ &= \begin{pmatrix} A^+\psi_n^+(x) \\ 0 \end{pmatrix} = \begin{pmatrix} \psi_{n+1}^-(x) \\ 0 \end{pmatrix} \end{aligned}$$





## Global Symmetries

Internal rigid symmetry groups are compact Lie groups space-time independents used in field or particles classification

For $g \in G$ and $\Phi$ a multiplet, we have

$$\Phi(x) \to U(g)\Phi(x)$$

explicitly in indices

$$\Phi_a(x) \to U_{ab}(g)\Phi_b(x)$$

with

$$(\Phi_a, \Phi_b) = \delta_{ab}$$

We say we have a Lagrangian symmetry when

$$L(U(g)\Phi) = L(\Phi)$$





To say that the transformation is rigid or global is to say that it does not depend on the space-time coordinates, i.e.,

$$\partial_\mu U(g)\Phi(x) = U(g)\partial_\mu \Phi(x)$$

Naturally, regarding the space-time, we assume the holding of Poincarè symmetry

$$(U(a,\Lambda)\Phi)(x) = D(\Lambda)\Phi(\Lambda^{-1}(x-a))$$

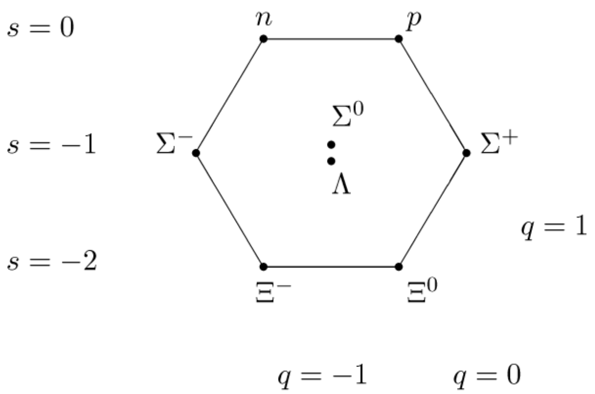
**Example**

$$\begin{aligned} L &= \bar{\psi}\partial\!\!\!/\psi + \tfrac{1}{2}(\partial_\mu\phi)^2 + V(\phi) \\ &+ \left[m_{\alpha\beta}\bar{\psi}_\alpha\psi_\beta + g^a_{\alpha\beta}\bar{\psi}_\alpha\psi_\beta\phi_a + f^k_{\alpha\beta}\bar{\psi}_\alpha\gamma_5\psi_\beta\psi_\beta\phi_k + h.c.\right] \end{aligned}$$

Even when the transformation

$$\Phi_a(x) \to U_{ab}(g)\Phi_b(x)$$

is not an exact Lagrangian symmetry, we may use it for classification purposes for instance as in the famous case of $SU(3)$-flavor (the eightfold way).








When the transformation

$$\Phi_a(x) \to U_{ab}(g)\Phi_b(x)$$

is a genuine symmetry, Noether's Theorem leads to the current conservatio

Indeed, define $I_k$ by

$$\left(\frac{\partial}{\partial a_k}(U(g)\Phi(x))^\alpha\right)_{a=0} = (I_k\Phi(x))^\alpha = (I_k^{\alpha\beta})\Phi^\beta(x)$$

and define the current

$$j_\mu^k = \pi_\mu I_k \Phi = \pi_\mu^\alpha I^{\alpha\beta}\Phi^\beta$$

where

$$\pi_\mu = \frac{\partial L(x)}{\partial \Phi_\mu(x)}, \qquad \Phi_\mu \equiv \partial_\mu \Phi$$





Therefore

$$\begin{aligned} \partial_\mu j_\mu^k &= (\partial_\mu \pi_\mu) I_k \Phi + \pi_\mu I_k(\partial_\mu \Phi) \\ &= \tfrac{\partial L}{\partial \Phi} I_k \Phi + \tfrac{\partial L}{\partial \Phi_\mu} I_k \Phi_\mu = \left(\tfrac{\partial L}{\partial a_k}\right)_{a=0} \end{aligned}$$

thus

$$\frac{\partial L}{\partial a_k} = 0 \Rightarrow \partial_\mu j_\mu^k = 0$$

and for each generator of the continuos group of symmetries we have a conserved current with corresponding charge

$$Q_k = \int d^3x j_0^k(x)$$

satisfying

$$\frac{\partial}{\partial t} Q_k = 0$$





## Local Symmetries

The gauge principle for electromagnetism

$$L_{EM} = L(\Phi, D_\mu \Phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$F_{i0} = E_i\,, \quad F_{ij} = \frac{1}{2}\epsilon_{ijk}B_k$$

The covariant devivative in this case reads

$$D_\mu \Phi = \partial_\mu \Phi + iA_\mu Q\Phi$$

with $Q\Phi_a = e_a\Phi_a$.

We have invariance with respect to gauge transformations

$$\Phi \to e^{i\theta(x)Q}\Phi\,,\ \ A_\mu \to A_\mu - \partial_\mu\theta(x)$$

Note that the potential $A_\mu$ is introduced in the covariant derivative precisely to get the covariance of the derivatives in the matter terms in such a way that

$$D_\mu(A^\theta)e^{i\theta(x)Q}\Phi(x) = e^{i\theta(x)Q}D_\mu(A)\Phi(x)$$

The idea now it to mimic this process to other Lie groups.





We begin with a matter Lagrangian, invariant with respect to a Lie group $G$.

$$L(\Phi(x), \partial_\mu\Phi(x)) = L(U(g)\Phi(x), U(g)\partial_\mu\Phi(x))$$

Introducing the vector potentials $A^k_\mu$, one corresponding to each generator $\sigma^k$ of $G$, we define the covariant derivative

$$D_\mu\Phi = \partial_\mu\Phi + e\,(A_\mu \cdot \sigma)\,\Phi$$

or

$$(D_\mu\Phi)^a = \partial_\mu\Phi^a + eA^k_\mu\sigma^{ab}_k\Phi^b$$

such that

$$U(g(x))D_\mu\Phi(x) = D_\mu U(g(x))\Phi(x)$$

or equivalently

$$D_\mu(A^g) = U(g(x))D_\mu(A)U^{-1}(g(x))$$





Developing the previous relation, we obtain the gauge field transformation

$$A^g_\mu = UA_\mu U^{-1} + U\partial_\mu U^{-1}$$

which generalizes the old ones

$$A\mu \to UA_\mu U^{-1} \qquad \text{e} \qquad A\mu \to A_\mu + \partial_\mu\theta$$

In thiw way, we obtain

$$L(\Phi(x), D_\mu\Phi(x)) = L(U(g(x))\Phi(x), U(g(x))D_\mu\Phi(x))$$





Similarly to the electromagnetic case, we construct now a kinetic term to the gauge fields and define

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + e[A_\mu, A_\nu]$$

from which we see

$$F_{\mu\nu}(A^g) = UF_{\mu\nu}(A)U^{-1}$$

That is, contrary to the Abelian case, the tensor $F_{\mu\nu}$ is not gauge invariant. But it is covariant, so that the Lagrangia

$$L(\Phi, A) = -\frac{1}{4}trF_{\mu\nu}F^{\mu\nu} + L(\Phi, D_\mu\Phi)$$

is invariant.

In order to discuss the gauge fixing condition let us first remind some basic properties of the maximal Abelian gauge in the case of $SU(2)$. The gauge field is decomposed into off-diagonal and diagonal components, according to

$$\mathcal{A}_\mu = A^a_\mu T^a + A_\mu T^3 , \tag{32}$$

where $T^a$, $a = 1, 2$, denote the off-diagonal generators of $SU(2)$, while $T^3$ stands for the diagonal generator,

$$\begin{aligned} \left[T^a, T^b\right] &= i\,\varepsilon^{ab} T^3, \\ \left[T^3, T^a\right] &= i\,\varepsilon^{ab} T^b, \end{aligned} \tag{33}$$

where

$$\begin{aligned} \varepsilon^{ab} &= \varepsilon^{ab3} , \\ \varepsilon^{ac}\varepsilon^{ad} &= \delta^{cd} . \end{aligned} \tag{34}$$





Similarly, for the field strength one has

$$\mathcal{F}_{\mu\nu} = F^a_{\mu\nu} T^a + F_{\mu\nu} T^3 , \tag{35}$$

with the off-diagonal and diagonal parts given, respectively, by

$$\begin{aligned} F^a_{\mu\nu} &= D^{ab}_\mu A^b_\nu - D^{ab}_\nu A^b_\mu , \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + g\varepsilon^{ab} A^a_\mu A^b_\nu , \end{aligned} \tag{36}$$

where the covariant derivative $D^{ab}_\mu$ is defined with respect to the diagonal component $A_\mu$

$$D^{ab}_\mu \equiv \partial_\mu \delta^{ab} - g\varepsilon^{ab} A_\mu . \tag{37}$$

Thus, for the Yang-Mills action in Euclidean space one obtains

$$S_{\rm YM} = \frac{1}{4}\int d^4x \, \left(F^a_{\mu\nu}F^a_{\mu\nu} + F_{\mu\nu}F_{\mu\nu}\right) . \tag{38}$$





As it is easily checked, the classical action (38) is left invariant by the gauge transformations

$$\begin{aligned} \delta A^a_\mu &= -D^{ab}_\mu \omega^b - g\varepsilon^{ab} A^b_\mu \omega , \\ \delta A_\mu &= -\partial_\mu \omega - g\varepsilon^{ab} A^a_\mu \omega^b . \end{aligned} \tag{39}$$

The maximal Abelian gauge is obtained by demanding that the off-diagonal components $A^a_\mu$ of the gauge field obey the nonlinear condition

$$D^{ab}_\mu A^b_\mu = 0 , \tag{40}$$

which follows by requiring that the auxiliary functional

$$\mathcal{R}[A] = \int d^4x A^a_\mu A^a_\mu , \tag{41}$$

is stationary with respect to the gauge transformations (39).





Moreover, as it is apparent from the presence of the covariant derivative $D^{ab}_\mu$, equation (40) allows for a residual local $U(1)$ invariance corresponding to the diagonal subgroup of $SU(2)$. This additional invariance has to be fixed by means of a suitable gauge condition on the diagonal component $A_\mu$, which will be chosen to be of the Landau type, also adopted in lattice simulations, namely

$$\partial_\mu A_\mu = 0 . \tag{42}$$

Let us work out the condition for the existence of Gribov copies in the maximal Abelian gauge. In the case of small gauge transformations, this is easily obtained by requiring that the transformed fields, eqs.(39), fulfill the same gauge conditions obeyed by $\left(A_\mu, A^a_\mu\right)$, *i.e.* eqs.(40), (42).

Thus, to the first order in the gauge parameters $(\omega, \omega^a)$, one gets

$$-D^{ab}_\mu D^{bc}_\mu \omega^c - g\varepsilon^{bc} D^{ab}_\mu \left(A^c_\mu \omega\right) \tag{43}$$
$$+g\varepsilon^{ab} A^b_\mu \partial_\mu \omega + g^2 \varepsilon^{ab}\varepsilon^{cd} A^b_\mu A^c_\mu \omega^d \;=\; 0\,, \tag{44}$$
$$-\partial^2\omega - g\varepsilon^{ab}\partial_\mu \left(A^a_\mu \omega^b\right) \;=\; 0\,, \tag{45}$$

which, due to eqs.(40),(42) read

$$\mathcal{M}^{ab}\omega^b \;=\; 0\,, \tag{46}$$
$$-\partial^2\omega - g\varepsilon^{ab}\partial_\mu \left(A^a_\mu \omega^b\right) \;=\; 0\,, \tag{47}$$

with $\mathcal{M}^{ab}$ given by

$$\mathcal{M}^{ab} = -D^{ac}_\mu D^{cb}_\mu - g^2\varepsilon^{ac}\varepsilon^{bd} A^c_\mu A^d_\mu\,. \tag{48}$$

The operator $\mathcal{M}^{ab}$ is recognized to be the Faddeev-Popov operator for the off-diagonal ghost sector.





It enjoys the property of being Hermitian and, is the difference of two positive semidefinite operators given, respectively, by $-D^{ac}_\mu D^{cb}_\mu$ and $g^2\varepsilon^{ac}\varepsilon^{bd} A^c_\mu A^d_\mu$. Also, one should remark that the diagonal parameter $\omega$ appears only in the eq.(47), in a form which allows us to express it in terms of the solution of the first equation (46). More precisely, once eq.(46) has been solved for $A_\mu$, $A^a_\mu$, $\omega^b$, for the diagonal parameter $\omega$ one can write

$$\omega = -g\epsilon^{ab}\frac{\partial_\mu}{\partial^2}\left(A^a_\mu \omega^b\right)\,. \tag{49}$$

This feature means essentially that the diagonal parameter $\omega$ has no special role in the characterization of the Gribov copies, whose properties are encoded in eq.(46).





Also, from eq.(49) it follows that the new variable $\tilde\omega$

$$\tilde\omega = \omega + g\epsilon^{ab}\frac{\partial_\mu}{\partial^2}\left(A^a_\mu \omega^b\right)\,, \tag{50}$$

obeys

$$\partial^2\tilde\omega = 0\,. \tag{51}$$

The change of variable (50) can be performed in the partition function expressing the Faddeev-Popov quantization of Yang-Mills theories in the maximal Abelian gauge. As the corresponding Jacobian turns out to be independent from the fields, transformation has the effect of decoupling the diagonal ghost fields from the theory. As a consequence, the corresponding two point function is not affected by the restriction to the Gribov region.





Let us face now the implementation in the Feynman path integral of the restriction of the domain of integration to the Gribov region $\mathcal{C}_0$, defined as the set of fields fulfilling the gauge conditions (40), (42) and for which the Faddeev-Popov operator $\mathcal{M}^{ab}$ is positive definite, namely

$$\begin{aligned}\mathcal{C}_0 \;=\; \{&A_\mu,\ A^a_\mu,\ \partial_\mu A_\mu = 0,\ D^{ab}_\mu A^b_\mu = 0,\\ &\mathcal{M}^{ab} = -D^{ac}_\mu D^{cb}_\mu - g^2\varepsilon^{ac}\varepsilon^{bd}A^c_\mu A^d_\mu > 0\}\,.\end{aligned} \tag{52}$$

The boundary, $l_1$, of the region $\mathcal{C}_0$, where the first vanishing eigenvalue of $\mathcal{M}^{ab}$ appears, is called the first Gribov horizon. The restriction of the domain of integration to this region is supported by the possibility of generalizing to the maximal Abelian gauge Gribov's original result stating that for any field located near a horizon there is a gauge copy, close to the same horizon, located on the other side of the horizon. We have found useful to devote the whole Appendix 9 to the details of the proof of this statement.

Thus, for the partition function of Yang-Mills theory in the maximal Abelian gauge, we write

$$\mathcal{Z} = \int DA^a_\mu DA_\mu \ \det\left(\mathcal{M}^{ab}(A)\right) \ \delta\left(D^{ab}_\mu A^b_\mu\right) \delta\left(\partial_\mu A_\mu\right) e^{-S_{YM}} \mathcal{V}(\mathcal{C}_0) \ , \tag{53}$$

where the factor $\mathcal{V}(\mathcal{C}_0)$ implements the restriction to the region $\mathcal{C}_0$. The factor $\mathcal{V}(\mathcal{C}_0)$ can be accommodated for by means of a no pole condition on the off-diagonal ghost two-point function, given by the inverse of the Faddeev-Popov operator $\mathcal{M}^{ab}$.





We are now ready to discuss the behavior of the gluon propagator when the domain of integration in the Feynman path integral is restricted to the region $\mathcal{C}_0$, eq.(53). The factor $\mathcal{V}(\mathcal{C}_0)$ implementing the restriction to $\mathcal{C}_0$ is given by

$$\mathcal{V}(\mathcal{C}_0) = \theta\left[1 - \sigma(0, A)\right] \ , \tag{54}$$

where $\theta(x)$ stands for the step function. Moreover, making use of the integral representation

$$\theta(1 - \sigma(0, A)) = \int_{-i\infty+\varepsilon}^{i\infty+\varepsilon} \frac{d\zeta}{2\pi i\zeta} e^{\zeta(1-\sigma(0,A))} \ , \tag{55}$$

for the partition function $\mathcal{Z}$ we get

$$\begin{aligned} \mathcal{Z} \ &= \ \int DA^a_\mu DA_\mu \tfrac{d\zeta}{2\pi i\zeta} \det\left(\mathcal{M}^{ab}(A)\right) \\ &\quad \exp\left(\zeta - S_{\mathrm{YM}} - \tfrac{1}{2\alpha}\left(D^{ab}_\mu A^b_\mu\right)^2 - \tfrac{1}{2\beta}\left(\partial_\mu A_\mu\right)^2 - \zeta\sigma(0, A)\right) \end{aligned} \tag{56}$$

where the gauge parameters $\alpha$ and $\beta$ have to be set to zero at end, *i.e.* $\alpha$, $\beta \to 0$, to recover the gauge conditions (40), (42).





In order to study the gluon propagator, it is sufficient to retain only the quadratic terms in expression (56) which contribute to the two-point correlation functions $\left\langle A^a_\mu(k) A^b_\nu(-k)\right\rangle$ and $\left\langle A_\mu(k) A_\nu(-k)\right\rangle$. Thus

$$\mathcal{Z}_{\mathrm{quadr}} = \mathcal{N} \int DA^a_\mu DA_\mu \frac{d\zeta}{2\pi i} e^{\left(\zeta - \log\zeta - S_{\mathrm{quadr}} - \zeta\sigma(0,A)\right)} \ , \tag{57}$$

where $\mathcal{N}$ is a constant factor and $S_{\mathrm{quadr}}$ stands for the quadratic part of the quantized Yang-Mills action, namely

$$\begin{aligned} S_{\mathrm{quadr}} \ &= \ \frac{1}{2}\sum_q \left(A^a_\mu(q)\left(q^2\delta_{\mu\nu} - \left(1 - \frac{1}{\alpha}\right) q_\mu q_\nu\right) A^a_\nu(-q)\right) \\ &\quad + \frac{1}{2}\sum_q \left(A_\mu(q)\left(q^2\delta_{\mu\nu} - \left(1 - \frac{1}{\beta}\right) q_\mu q_\nu\right) A_\nu(-q)\right) \end{aligned} \tag{58}$$





Taking the thermodynamic limit, $V \to \infty$, and introducing the Gribov parameter $\gamma$

$$\gamma^4 = \frac{\zeta_0 g^2}{2V} \ , \qquad V \to \infty \ , \tag{59}$$

we get the gap equation

$$\frac{3}{4} g^2 \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^4 + \gamma^4} = 1 \ , \tag{60}$$

where the term $1/\zeta_0$ has been neglected in the thermodynamic limit. To obtain the gauge propagator, we can now go back to the expression for $\mathcal{Z}_{\mathrm{quadr}}$ which, after substituting the saddle point value $\zeta = \zeta_0$, becomes

$$\mathcal{Z}_{\mathrm{quadr}} = \mathcal{N} \int DA^a_\mu DA_\mu e^{-\frac{1}{2}\left(\sum_q A_\mu(q)\mathcal{Q}_{\mu\nu}(\gamma,q)A_\nu(-q) + \sum_q A^a_\mu(q)\mathcal{P}_{\mu\nu}(q)A^a_\mu(-q)\right)} \ , \tag{61}$$

with

$$\mathcal{Q}_{\mu\nu}(\gamma, q) = \left(q^2 + \frac{\gamma^4}{q^2}\right)\delta_{\mu\nu} - \left(1 - \frac{1}{\beta}\right) q_\mu q_\nu \ . \tag{62}$$

Evaluating the inverse of $\mathcal{Q}_{\mu\nu}(\gamma, q)$ and of $\mathcal{P}_{\mu\nu}(q)$, and setting the gauge parameters $\alpha$,$\beta$ to zero, we get the gluon propagator for the diagonal and off diagonal components of the gauge field, namely

$$\langle A_\mu(q)A_\nu(-q)\rangle = \frac{q^2}{q^4+\gamma^4}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) , \tag{63}$$

and

$$\left\langle A^a_\mu(q)A^b_\nu(-q)\right\rangle = \delta^{ab}\frac{1}{q^2}\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) , \tag{64}$$

One sees that the diagonal component, eq.(63), is suppressed in the infrared, exhibiting the characteristic Gribov type behavior. The off-diagonal components, eq.(64), remains unchanged.





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