# III_ APPLICATIONS OF QUANTUM MECHANICS

## 1_ ANGULAR MOMENTUM

### Motivation

In classical mechanics, the ordinary angular momentum three-vector is defined as

$$\vec{L} = \vec{x} \times \vec{p} = (x_2 p_3 - x_3 p_2)\hat{e}_1 + (x_3 p_1 - x_1 p_3)\hat{e}_2 + (x_1 p_2 - x_2 p_1)\hat{e}_3$$

In QM, inspired by the classical analog, we define the <u>orbital angular momentum</u> operator as

$$\begin{cases} L_1 = X_2 P_3 - X_3 P_2 \\ L_2 = X_3 P_1 - X_1 P_3 \\ L_3 = X_1 P_2 - X_2 P_1 \end{cases}$$

which of course is the same as $\vec{L} = \vec{X} \times \vec{P}$

or in index notation simply

$$\boxed{L_i = \epsilon_{ijk} X_j P_k}$$

Note that in the QM context $X_i$ and $P_i$ are operators. Since $[X_i, P_j] = i\hbar\delta_{ij}$ we do not have commutation issues in the definition of $L_i$ above. For instance in the definition of $L_1$, the operators $X_2$ and $P_3$ commute with each other.

From the canonical commutation relation

$$[X_i, P_j] = i\hbar\,\delta_{ij}$$

we immediately obtain

$$\begin{cases} (i) & [L_i, P_j] = i\hbar\,\epsilon_{ijk} P_k \\ (ii) & [L_i, X_j] = i\hbar\,\epsilon_{ijk} X_k \\ (iii) & [L_i, L_j] = i\hbar\,\epsilon_{ijk} L_k \end{cases}$$

In fact

$$[L_i, P_j] = [\epsilon_{ik\ell} X_k P_\ell, P_j]$$

$$= \epsilon_{ik\ell}\, i\hbar\,\delta_{kj} P_\ell = i\hbar\,\epsilon_{ijk} P_k$$

$$[L_i, X_j] = [\epsilon_{ik\ell} X_k P_\ell, X_j]$$

$$= \epsilon_{ik\ell} X_k (-i\hbar\,\delta_{\ell j})$$

$$= -i\hbar\,\epsilon_{ikj} X_k = i\hbar\,\epsilon_{ijk} X_k$$

$$[L_i, L_j] = \epsilon_{iab}\epsilon_{jcd}[X_a P_b, X_c P_d]$$

$$= \epsilon_{iab}\epsilon_{jcd}\{X_c\, i\hbar\,\delta_{ad} P_b - X_a\, i\hbar\,\delta_{bc} P_d\}$$

$$= i\hbar\{\epsilon_{ikb}\epsilon_{jak} - \epsilon_{iak}\epsilon_{jkb}\} X_a P_b$$

$$= i\hbar\{-\epsilon_{ibk}\epsilon_{jak} + \epsilon_{iak}\epsilon_{bjk}\} X_a P_b$$

$$= i\hbar\{-\epsilon_{ibk}\epsilon_{jak} - \epsilon_{iak}\epsilon_{bjk}\} X_a P_b$$

$$[L_i, L_j] = i\hbar \{ -\epsilon_{ibk}\epsilon_{jak} - \epsilon_{iak}\epsilon_{bjk} \} X_a P_b$$

$$= i\hbar \{ \epsilon_{ijk}\epsilon_{abk} \} X_a P_b$$

$$= i\hbar\, \epsilon_{ijk} \{ \epsilon_{kab} X_a P_b \}$$

$$= i\hbar\, \epsilon_{ijk} L_k$$

The relation

$$[L_i, L_j] = i\hbar\, \epsilon_{ijk} L_k$$

is very remarkable and actually characterizes the Lie algebras of $so(3)$ and $su(2)$.

**Definition**: Any three operators $J_1$, $J_2$ and $J_3$ satisfying the commutation relations

$$[J_i, J_j] = i\hbar\, \epsilon_{ijk} J_k$$

are called <u>angular momentum operators</u>.

<u>Rem</u>: Assume we have three operators $A_i$, $i=1,2,3$, satisfying

$$[A_i, A_j] = \alpha\, \epsilon_{ijk} A_k .$$

Then we can always redefine

$$J_i \equiv \frac{i\hbar}{\alpha} A_i$$

to get

$$[J_i, J_j] = \left[\frac{i\hbar}{\alpha} A_i, \frac{i\hbar}{\alpha} A_j\right]$$

$$= \left(\frac{i\hbar}{\alpha}\right)^2 [A_i, A_j]$$

$$= \left(\frac{i\hbar}{\alpha}\right)^2 \alpha\, \epsilon_{ijk} A_k$$

$$= i\hbar\, \epsilon_{ijk} \underbrace{\frac{i\hbar A_k}{\alpha}}_{J_k} = i\hbar\, \epsilon_{ijk} J_k$$

Given an angular momentum operator

$$\vec{J} = J_i \hat{e}_i \,, \quad [J_i, J_j] = \epsilon_{ijk}\, i\hbar\, J_k$$

for two arbitrary three-vectors. $\vec{a}, \vec{b} \in \mathbb{R}^3$

we form the products

$$\vec{a} \cdot \vec{J} = a_i J_i \,, \quad \vec{b} \cdot \vec{J} = b_i J_i$$

to have the identity

$$[\vec{a} \cdot \vec{J}, \vec{b} \cdot \vec{J}] = i\hbar\, (\vec{a} \times \vec{b}) \cdot \vec{J}$$

In fact, since $\vec{a}$ and $\vec{b}$ are not operators and always commute, we have

$$[\vec{a} \cdot \vec{J}, \vec{b} \cdot \vec{J}] = a_i b_j [J_i, J_j]$$

$$= i\hbar\, \epsilon_{ijk}\, a_i b_j J_k = i\hbar\, (\vec{a} \times \vec{b})_k J_k$$

$$= i\hbar (\vec{a} \times \vec{b}) \cdot \vec{J}$$

# More Commutation Relations

Given three operators $J_i$, $i = 1, 2, 3$, satisfying

$$[J_i, J_j] = i\hbar \epsilon_{ijk} J_k$$

(i.e., given an angular momentum operator set $J_i$), we define the <u>total angular momentum</u> as

$$J^2 = J_1^2 + J_2^2 + J_3^2 = J_i J_i$$

Exercise: Show that

$$[J_i, J^2] = 0$$

solution:

$$[J_i, J_k J_k] = J_k [J_i, J_k] + [J_i, J_k] J_k$$

$$= i\hbar \epsilon_{ik\ell} J_k J_\ell + i\hbar \epsilon_{ik\ell} J_\ell J_k = 0$$

Therefore, the total angular momentum $J^2$ commutes with each of its three components $J_i$. Actually, we may prove the following more general

<u>Proposition</u>: If $[J_i, A_j] = i\hbar \epsilon_{ijk} A_k$ and $[J_i, B_j] = i\hbar \epsilon_{ijk} B_k$, then

$$[J_i, \vec{A} \cdot \vec{B}] = 0$$

Proof: By direct computation

$$[J_i, A_j B_j] = A_j [J_i, B_j] + [J_i, A_j] B_j$$

$$= i\hbar\, \epsilon_{ijk} A_j B_k + i\hbar\, \epsilon_{ijk} A_k B_j$$

$$= i\hbar \left( \epsilon_{ijk} + \epsilon_{ikj} \right) A_j B_k = 0$$

Corollary:

(i) For general angular momentum $J_i$, we have

$$[J^2, J_i] = 0, \qquad i = 1, 2, 3$$

(ii) For orbital angular momentum $\vec{L} = \vec{X} \times \vec{P}$

we have

$$\begin{cases} [L^2, L_i] = 0 & (A_i = B_i = L_i) \\ [P^2, L_i] = 0 & (A_i = B_i = P_i) \\ [X^2, L_i] = 0 & (A_i = B_i = X_i) \\ [\vec{X}.\vec{P}, L_i] = 0 & (A_i = X_i,\ B_i = P_i) \\ [\vec{X}.\vec{L}, L_i] = 0 & (A_i = X_i,\ B_i = L_i) \\ [\vec{P}.\vec{L}, L_i] = 0 & (A_i = P_i,\ B_i = L_i) \end{cases}$$

# Ladder Operators

Given an angular momentum operator set $J_i$, $[J_i, J_j] = i\hbar \epsilon_{ijk} J_k$, we define the corresponding <u>ladder operators</u> $J_\pm$ as

$$J_\pm = J_1 \pm i J_2$$

<u>Proposition</u>: The ladder operators satisfy

(i) $[J^2, J_\pm] = 0$

(ii) $J_\mp J_\pm = J^2 - (J_3^2 \pm \hbar J_3)$

(iii) $[J_+, J_-] = 2\hbar J_3$

(iv) $[J_3, J_\pm] = \pm \hbar J_\pm$

<u>Proof</u>:

(i) $[J^2, J_\pm] = [J^2, J_1 \pm i J_2] = 0$

(ii) $J_\mp J_\pm = (J_1 \mp i J_2)(J_1 \pm i J_2)$

$= J_1^2 + J_2^2 \pm i [J_1, J_2]$

$= J^2 - J_3^2 \mp \hbar J_3 = J^2 - (J_3^2 \pm \hbar J_3)$

(iii) $[J_+, J_-] = [J_1 + i J_2, J_1 - i J_2]$

$= -i [J_1, J_2] + i [J_2, J_1] = 2\hbar J_3$

(iv) $[J_3, J_\pm] = [J_3, J_1 \pm i J_2] = [J_3, J_1] \pm i [J_3, J_2]$

<u>corollary</u>: Let $\psi$ be a common $J^2$ and $J_3$ eigenvector with $J^2\psi = \lambda\hbar^2\psi$ and $J_3\psi = m\hbar\psi$. Then

(i) $J^2 J_\pm \psi = \lambda \hbar^2 J_\pm \psi$

(ii) $J_3 J_\pm \psi = (m \pm 1)\hbar J_\pm \psi$

(iii) $\| J_\pm \psi \|^2 = [\lambda - m(m \pm 1)]\hbar^2 \|\psi\|^2$

(iv) $\lambda \geqslant m(m \pm 1)$

(v) $\lambda = m(m \pm 1)$ iff $J_\pm \psi = 0$

<u>proof</u>:

(i) Since $[J^2, J_\pm] = 0$ we have $J^2(J_\pm\psi) = J_\pm(J^2\psi) = J_\pm \hbar^2 \lambda \psi$

(ii)

$$J_3 J_\pm \psi = \Big\{ \underbrace{[J_3, J_\pm]}_{\pm\hbar J_\pm} + J_\pm J_3 \Big\} \psi$$

$$= (\pm\hbar J_\pm + m\hbar J_\pm)\psi = (m \pm 1)\hbar J_\pm \psi$$

(iii)

$$\| J_\pm \psi \|^2 = \langle J_\pm \psi | J_\pm \psi \rangle$$

$$= \langle \psi | J_\mp J_\pm \psi \rangle$$

$$= \langle \psi | \{ J^2 - (J_3^2 \pm \hbar J_3) \} \psi \rangle = (\lambda^2 - m(m \pm 1))\hbar^2 \|\psi\|^2$$

(iv) Since $\| J_\pm \psi \|^2 \geqslant 0$ we have $\lambda^2 - m(m \pm 1) \geqslant 0$

(v) $J_\pm \psi = 0 \Leftrightarrow \| J_\pm \psi \|^2 = 0 \Leftrightarrow \lambda^2 - m(m \pm 1) = 0$

**Proposition**: The eigenvalues of the total angular momentum operator $J^2$ are $j(j+1)\hbar^2$ with $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

For each semi-integer $j$ the corresponding eigenvalues of $J_3$ are $m\hbar$ with $m = -j, -j+1, \ldots, j-1, j$.

**Proof**: Consider $\psi \in \mathcal{H}$ such that $J^2\psi = \lambda \hbar^2 \psi$ and $J_3\psi = m\hbar\psi$ (this is possible because $[J_3, J^2] = 0$).

From $\lambda \geq m(m \pm 1)$ we have

$$\lambda + \frac{1}{4} \geq m^2 \pm m + \frac{1}{4} = (m \pm \tfrac{1}{2})^2 .$$

That means, for a given $\lambda$, there exists a maximal and a minimal value for $m$. Note however that for a fixed given $\lambda$, application of the operators $J_\pm$ in $\psi$ raises and lowers $m$ without affecting $j$:

$$J_3(J_+\psi) = (m+1)\hbar J_+\psi$$

$$\begin{aligned} J_3(J_+^2\psi) &= \{ [J_3, J_+] + J_+ J_3 \} J_+\psi \\ &= \{ \hbar J_+ + J_+(m+1)\hbar \} J_+\psi \\ &= \hbar(m+2) J_+^2 \psi \end{aligned}$$

$$J_3(J_+^N\psi) = \hbar(m+N)(J_+^N\psi)$$

Thus there must exist a minimal natural $p \in \mathbb{N}$ such that

$$J_+^{p+1}\psi = 0 \quad \text{with} \quad J_+^p\psi \neq 0$$

and similarly from $J_-$, a minimal natural $q \in \mathbb{N}$ such that

$$J_-^{q+1}\psi = 0 \quad \text{with} \quad J_-^q\psi \neq 0.$$

That means we have

$$\begin{cases} J^2\psi = \lambda\hbar^2\psi \\ J_3\psi = m\hbar\psi \end{cases} \qquad \begin{cases} J^2 J_+^p\psi = \lambda\hbar^2 J_+^p\psi \\ J_3 \underbrace{J_+^p\psi}_{M=m+p} = (m+p)\hbar \underbrace{J_+^p\psi}_{M=m+p} \end{cases} \qquad \begin{cases} J_+^p\psi \neq 0 \\ J_+^{p+1}\psi = 0 \end{cases}$$

recall

$$J_+\big(\underbrace{J_+^p\psi}_{M=m+p}\big) = 0 \iff \lambda = M(M+1) = (m+p)(m+p+1)$$

$$\boxed{\lambda = (m+p)(m+p+1)}$$

and similarly for the $J_-$ case:

$$\begin{cases} J^2\psi = \lambda\hbar^2\psi \\ J_3\psi = m\hbar\psi \end{cases} \xrightarrow{J_-^q} \begin{cases} J^2 J_-^q\psi = \lambda\hbar^2 J_-^q\psi \\ J_3 J_-^q\psi = (m-q)\hbar J_-^q\psi \end{cases} \qquad \begin{cases} J_-^q\psi \neq 0 \\ J_-^{q+1}\psi = 0 \end{cases}$$

$$J_-\big(\underbrace{J_-^q\psi}_{M=m-q}\big) = 0 \iff \lambda = M(M-1) = (m-q)(m-q-1)$$

$$\boxed{\lambda = (m-q)(m-q-1)}$$

Of course the two results for $\lambda$ must be equal