The Bohm - Aharonov Effect

Recall the Lorentz force

$$\vec{F} = e\vec{E} + e\vec{v} \times \vec{B}$$

with

$$\begin{cases} \vec{E} = -\dfrac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi \\ \vec{B} = \vec{\nabla} \times \vec{A} \end{cases}$$

The potentials $\vec{A}$ and ϕ are not unique, given an arbitrary real function

$$\chi(\vec{r}, t)$$

the potentials

$$\vec{A}' = \vec{A} - \vec{\nabla}\chi$$

$$\phi' = \phi + \frac{\partial \chi}{\partial t}$$

leads to the same electromagnetic fields $\vec{E}, \vec{B}$.

Indeed

$$\vec{E}' = -\frac{\partial \vec{A}}{\partial t} + \frac{\partial \vec{\nabla} \chi}{\partial t} - \vec{\nabla}\phi - \vec{\nabla}\frac{\partial \chi}{\partial t}$$

$$= -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi + \vec{\nabla}\left(\frac{\partial \chi}{\partial t}\right) - \vec{\nabla}\left(\frac{\partial \chi}{\partial t}\right)$$

$$= -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi = \vec{E}$$

$$\vec{B}' = \vec{\nabla}\times\vec{A} - \underbrace{\vec{\nabla}\times(\vec{\nabla}\chi)}_{\vec{0}}$$

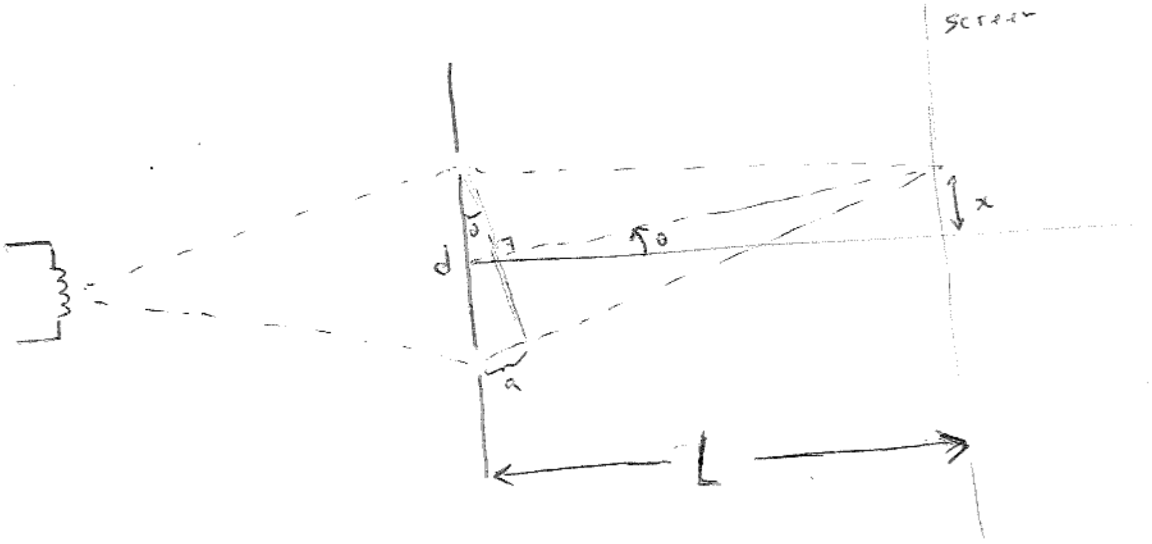
$$= \vec{\nabla}\times\vec{A} = \vec{B}$$

Rem.:

$$e_{ijk}\,\partial_j\,(\vec{\nabla}\chi)_k\,\hat{e}_i =$$

$$= \underbrace{e_{ijk}}_{\text{ant}}\,\underbrace{\partial_j\,\partial_k}_{\text{sym.}}\,\chi\,\hat{e}_i = 0$$

Consider now the well known 2-slit experiment with an electron beam



By considering $x, d << L$ we use the approximation

$$tg\,\theta = \frac{x}{L} \cong \frac{a}{d}$$

So that $a \cong \left(\frac{x}{L}\right) d$

Considering plane waves for the electron beam, the path difference is a and we have maxima when

$$\frac{a}{\lambda} = m$$

where λ is the wave length (the path difference must be a multiple of the wave length).

That is

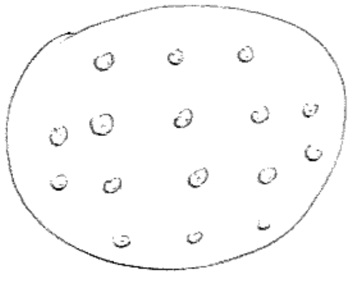
$$\frac{x\,d}{\lambda L} = m \quad , \quad \boxed{x = m \frac{\lambda}{d} L}$$

The maxima occur on the screen for

$$x = m \frac{\lambda}{d} L \qquad \left(x = 2\pi\, m \frac{\lambda L}{2\pi\, d} = \frac{\delta\, \lambda L}{2\pi\, d} \right)$$

while the minima occur in between.

Now we introduce a solenoid behind the wall between the slits



Given a general vector $\vec{A}$, its rotational in cilyndrical coordinates may be written as

$$\vec{\nabla} \times \vec{A} = \left(\frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \hat{r} +$$

$$+ \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \hat{\phi} + \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_\phi) - \frac{\partial A_r}{\partial \phi} \right] \hat{z}$$

Since we need $\vec{B} = B\hat{z}$, from $\vec{B} = \vec{\nabla} \times \vec{A}$ we see we must have, for $r < R$.

$$B\hat{z} = \left(\frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \hat{r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \hat{\theta} +$$

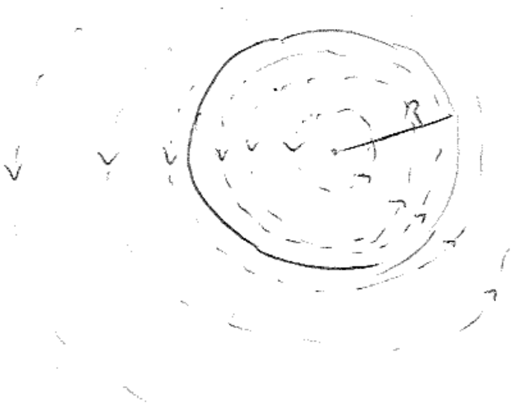
$$+ \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_\phi) - \frac{\partial A_r}{\partial \phi} \right] \hat{z}$$

One possible solution is

$$A_r = A_z = 0 \ , \qquad A_\phi = \frac{Br}{2} \ , \quad r < R$$

For $r \gg R$, we must have $\vec{B} = \vec{0}$. Maintaining continuity for $r = R$, we may write

$$A_r = A_z = 0 \ , \qquad A_\phi = \frac{BR^2}{2r}$$



The wave function for the free electron field is given by the plane wave

$$\psi = C \exp\left(\frac{i}{\hbar} \vec{p} \cdot \vec{r}\right)$$

$$= C \exp(i\alpha)$$

with $\alpha = \frac{\vec{p}\cdot\vec{r}}{\hbar}$ a phase.

We know the effect of turning on the electromagnetic field is given by

$$\vec{p} \longrightarrow \vec{p} - e\vec{A}$$

which amounts in changing

$$\alpha \longrightarrow \frac{\vec{p}\cdot\vec{r}}{\hbar} - \frac{e\vec{A}\cdot\vec{r}}{\hbar}$$

The change in phase over an entire trajectory is

$$\Delta\alpha = -\frac{e}{\hbar} \int \vec{A}\cdot\vec{r}$$

Considering trajectories 1 and 2, we have

$$\Delta\alpha_1 = -\frac{e}{\hbar}\int_1 \vec{A}\cdot d\vec{r} \qquad \Delta\alpha_2 = -\frac{e}{\hbar}\int_2 \vec{A}\cdot d\vec{r}$$

$$\Delta\delta = \Delta\alpha_1 - \Delta\alpha_2$$

$$= -\frac{e}{\hbar}\int_1 \vec{A}\cdot d\vec{r} + \frac{e}{\hbar}\int_2 \vec{A}\cdot d\vec{r}$$

$$= \frac{e}{\hbar}\oint \vec{A}\cdot d\vec{r}$$

$$= \frac{e}{\hbar}\int \left(\vec{\nabla}\times\vec{A}\right)\cdot \hat{n}\, dS$$

$$= \frac{e}{\hbar}\int \vec{B}\cdot\hat{n}\, dS$$

$$= \frac{e}{\hbar}\Phi$$

The interference pattern changes by

$$\Delta x = \Delta\delta\,\frac{\lambda L}{2\pi d} = \frac{e}{\hbar}\,\frac{\lambda L\,\Phi}{2\pi d}$$

Considering the region $r > R$, we have $\vec{B} = \vec{\nabla} \times \vec{A} = \vec{0}$, as we already saw

$$A_\phi = \frac{BR^2}{2r} \quad , \quad A_r = A_z = 0$$

From $\vec{A} = \vec{\nabla}\chi$, we have

$$A_\phi = \frac{1}{r}\frac{\partial \chi}{\partial \phi} = \frac{BR^2}{2r}$$

$$\chi = \frac{BR^2\phi}{2}$$

and χ is not a "single-valued" in space function

$$\Delta\delta = \Delta\alpha_1 - \Delta\alpha_2$$

$$= \frac{e}{\hbar}\oint \vec{A}\cdot d\vec{r}$$

$$= \frac{e}{\hbar}\oint (\vec{\nabla}\chi)\cdot d\vec{r} = \frac{e}{\hbar}\chi\Big|_0^{2\pi} = \frac{e}{\hbar}\frac{BR^2 2\pi}{2}$$

$$= \frac{e}{\hbar}\Phi$$

$$= 5\,Y =$$

We know that if $\vec{A} = \vec{\nabla}\chi$ everywhere then we would have $\vec{B} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times(\vec{\nabla}\chi) = \vec{0}$ everywhere. But that is not possible.

We consider now the problem defined only in the region $r \geq R$ with $R > 0$. Then we may have

$$\begin{cases} \vec{A} = \vec{\nabla}\chi, & r \geq R \\ \vec{B} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times(\vec{\nabla}\chi) = \vec{0}, & r \geq R \end{cases}$$

In fact we have seen we may take in cylindrical coordinates

$$\chi(r, \phi, z) = \frac{BR^2\phi}{2}$$

this leads to

$$\vec{A} = \hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z}$$

$$\vec{A} = \frac{BR^2}{2r}\hat{\phi}$$

$$\left(A_r = A_z = 0, \quad A_\phi = \frac{BR^2}{2r}\right)$$

and

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad , \; r \geq R$$

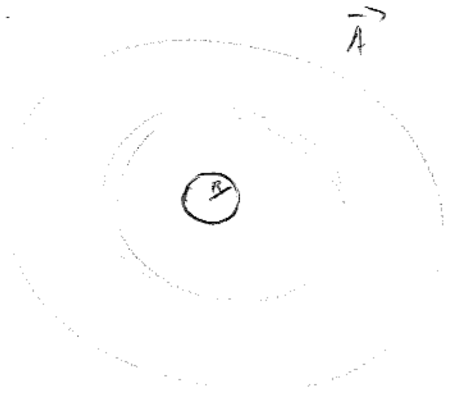
$$= \left(\frac{1}{r} \cancelto{0}{\frac{\partial A_z}{\partial \phi}} - \frac{\partial A_\phi}{\partial z} \right) \hat{r} \; +$$

$$+ \left(\cancelto{0}{\frac{\partial A_r}{\partial z}} - \cancelto{0}{\frac{\partial A_z}{\partial r}} \right) \hat{\phi} + \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_\phi) - \cancelto{0}{\frac{\partial A_r}{\partial \phi}} \right] \hat{z}$$

$$= -\frac{\partial}{\partial z} \left(\frac{BR^2}{2r} \right) \hat{r} + \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{BR^2}{2} \right) \hat{z}$$

$$= \vec{0} \qquad , \; r \geq R$$

That is, $\vec{B} \equiv \vec{0}$ for $r \geq R$



Now observe that $\chi(r, \phi, z) = \chi(\phi) = \frac{BR^2\phi}{2}$

is not a single-valued function:

We have already calculated the change in phase for the electron as

$$\Delta\delta = \Delta\alpha_1 + \Delta\alpha_2$$

$$= \frac{e}{\hbar}\oint \vec{A}\cdot d\vec{r}$$

for $r < R$

$$= \frac{e}{\hbar}\int(\vec{\nabla}\times\vec{A})\cdot\hat{n}\, dS$$

for $r < R$

$$= \frac{e}{\hbar}\int \vec{B}\cdot\hat{n}\, dS$$

$$= \frac{e}{\hbar}\Phi$$

where Φ was the magnetic flux. Since we want to consider now only the region $r > R$, we must have for consistency:

$$\frac{e}{\hbar}\Phi = \Delta\delta = \frac{e}{\hbar}\oint\vec{A}\cdot d\vec{r} = \frac{e}{\hbar}\oint\vec{\nabla}\chi\cdot d\vec{r}$$

$$= \frac{e}{\hbar}\chi\Big|_0^{2\pi} = \frac{e}{\hbar}\frac{BR^2 \cancel{2}\pi}{\cancel{2}}$$

Thus we see, for consistency, χ is not single-valued in the sense that the value of the integral

$$\oint \vec{A} \cdot d\vec{r}$$

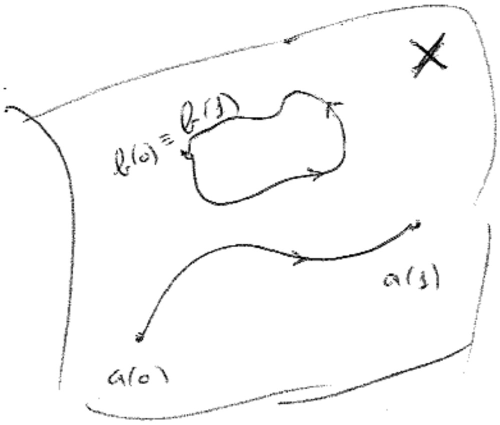
depends whether the path passes through the role, in which ~~same~~ direction (clockwise or counterclockwise) and depends on how many times it circles through.

That leads us to the interesting subject of <u>topology</u>.

A path a in a topological space X is a continuous function $a(s)$ of a real parameter s. We may restrict this definition to let the parameter s vary in the interval $[0,1]$

$$a : [0,1] \longrightarrow X$$

We may have both closed or open paths



Two paths a and b with the same initial and final points ($x_0 = a(0) = b(0)$, $x_1 = a(1) = b(1)$) are said to be <u>homotopic</u> when there exists a continuous function $L(t,s)$ such that

$$L(0,s) = a(s)$$
$$L(1,s) = b(s)$$

Path homotopy is an equivalence relation.

In this case we write $a \sim b$

Given a path $a : [0,1] \to X$ we define its inverse $\bar{a}^1 : [0,1] \to X$ by

$$\bar{a}^1(s) = a(1-s)$$

Given two paths a and b with $a(1) = b(0)$ we define the product path $c = ab$ by

$$c(s) = \begin{cases} a(2s), & 0 \leq s \leq \frac{1}{2} \\ b(2s-1), & \frac{1}{2} < s \leq 1 \end{cases}$$

The product of curves induces a product in the set of equivalence classes of curves in X (this product is not always ~~...~~ defined).

<u>Def.</u>: Let X be a top space and $x_0 \in X$. A path which begins and ends at x_0 is a <u>loop based on x_0</u>. The set of homotopy classes of loops based on x_0 constitutes a group under the previously defined product. This is the fundamental group of X relative to base point x_0, denoted by $\pi_1(X, x_0)$.

The Yang-Mills Field

We recall we had a Lagrangian density

$$\mathcal{L} = \partial_\mu \phi \, \partial^\mu \phi^* - m^2 \phi^* \phi$$

which could be rewritten as

$$\mathcal{L} = (\partial_\mu \underset{\sim}{\phi}) \cdot (\partial^\mu \underset{\sim}{\phi}) - m^2 \underset{\sim}{\phi} \cdot \underset{\sim}{\phi}$$

with

$$\phi = (\phi_1 + i\phi_2)/\sqrt{2}$$

$$\phi^* = (\phi_1 - i\phi_2)/\sqrt{2}$$

and

$$\underset{\sim}{\phi} = \hat{i}\phi_1 + \hat{j}\phi_2$$

and which was globally invariant under SO(2)

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos\Lambda & \sin\Lambda \\ -\sin\Lambda & \cos\Lambda \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$$\phi' = e^{-i\Lambda}\phi$$

$$\phi^{*\prime} = e^{i\Lambda}\phi$$

By introducing the gauge field A_μ we have obtained the total Lagrangian density

$$\mathcal{L}_3 = (D_\mu \phi)(D^\mu \phi^*) - m^2 \phi^* \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

with $D_\mu \equiv \partial_\mu + ieA_\mu$

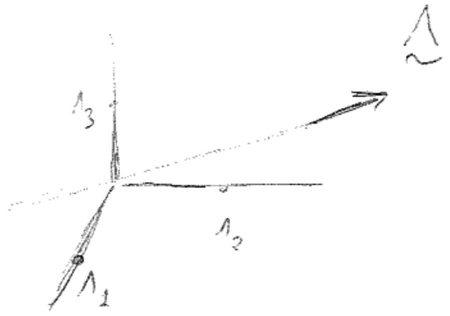
which is locally invariant under either

finite transfs:	infinitesimal transfs:
$A_\mu \to A_\mu' = A_\mu + \frac{1}{e}\partial_\mu \Lambda$	$\delta A_\mu = \frac{1}{e}\partial_\mu \Lambda$
$\phi \to \phi' = e^{-i\Lambda}\phi$	$\delta\phi = -i\Lambda\phi$
$\phi^* \to \phi^{*\prime} = e^{i\Lambda}\phi^*$	$\delta\phi^* = i\Lambda\phi^*$

The idea now is to generalize from the group $U(1) \cong SO(2)$ to $SO(3)$.

Consider an internal space, three dimensional, for the field ϕ. An infinitesimal rotation may be described by

$$\underset{\sim}{\Lambda} = \Lambda_1 \hat{e}_1 + \Lambda_2 \hat{e}_2 + \Lambda_3 \hat{e}_3$$



Since the rotation is infinitesimal,

$$\left(1 + \frac{i}{\hbar} \Lambda_1 L_1\right)\left(1 + \frac{i}{\hbar} \Lambda_2 L_2\right)\left(1 + \frac{i}{\hbar} \Lambda_3 L_3\right) =$$

$$= \left(1 + \frac{i}{\hbar} \Lambda_k L_k\right)$$

The field $\underset{\sim}{\phi} = \phi_k \hat{e}_k$ transforms as

$$\underset{\sim}{\phi} \longrightarrow \underset{\sim}{\phi}' = \exp\left[1 + \frac{i}{\hbar} \Lambda_k L_k\right] \underset{\sim}{\phi}$$

$$= \underset{\sim}{\phi} + \frac{i}{\hbar} \Lambda_k L_k \underset{\sim}{\phi} .$$

Thus, in components we have

$$\phi_j' = \phi_j + \frac{i}{\hbar} \Lambda_k \left(L_k \underset{\sim}{\phi} \right)_j$$

$$= \phi_j + \frac{i}{\hbar} \Lambda_k \left(L_k \right)_{ji} \phi_i$$

$$= \phi_j + \frac{i}{\hbar} \Lambda_k \left(- i \hbar \epsilon_{kji} \right) \phi_i$$

$$= \phi_j + \epsilon_{kji} \Lambda_k \phi_i$$

$$= \phi_j - \epsilon_{jki} \Lambda_k \phi_i$$

$$= \phi_j - \left(\underset{\sim}{\Lambda} \times \underset{\sim}{\phi} \right)_j$$

A bit more compact, this can be rewritten as

$$\underset{\sim}{\phi} \longrightarrow \underset{\sim}{\phi}' = \underset{\sim}{\phi} - \underset{\sim}{\Lambda} \times \underset{\sim}{\phi}$$

Since the infinitesimal parameter $\underset{\sim}{\Lambda}$ does not depend on space time we have a <u>global symmetry</u> (or gauge transf. of the first kind).

The group of symmetries is $SO(3)$ and look forward in generalizing it to a local gauge transf.

Note that, if we let $\underset{\sim}{\Lambda}$ depend on space time, we have

$$\underset{\sim}{\phi} \to \underset{\sim}{\phi}' = \underset{\sim}{\phi} - \underset{\sim}{\Lambda} \times \underset{\sim}{\phi}$$

$$\partial_\mu \underset{\sim}{\phi} \to \partial_\mu \underset{\sim}{\phi}' = \partial_\mu \underset{\sim}{\phi} - \Lambda \times \partial_\mu \underset{\sim}{\phi} - \partial_\mu \Lambda \times \underset{\sim}{\phi}$$

which means

$$\delta(\partial_\mu \underset{\sim}{\phi}) = - \Lambda \times \partial_\mu \underset{\sim}{\phi} - \partial_\mu \Lambda \times \underset{\sim}{\phi}$$

and $\partial_\mu \underset{\sim}{\phi}$ does not transform covariantly.

We introduce a new gauge field $\underset{\sim}{W}_\mu$ and the covariant derivative

$$D_\mu \underset{\sim}{\phi} = \partial_\mu \underset{\sim}{\phi} + g \underset{\sim}{W}_\mu \times \underset{\sim}{\phi}$$

requiring that

$$\delta(D_\mu \underset{\sim}{\phi}) = - \underset{\sim}{\Lambda} \times (D_\mu \underset{\sim}{\phi})$$

So, on one hand we have

$$\delta(D_\mu \underset{\sim}{\phi}) = -\underset{\sim}{\Lambda} \times D_\mu \underset{\sim}{\phi}$$

and on the other hand

$$\delta(D_\mu \underset{\sim}{\phi}) = \delta\left[\partial_\mu \underset{\sim}{\phi} + g \underset{\sim}{W}_\mu \times \underset{\sim}{\phi}\right]$$

$$= \delta(\partial_\mu \underset{\sim}{\phi}) + g(\delta \underset{\sim}{W}_\mu) \times \underset{\sim}{\phi} + g \underset{\sim}{W}_\mu \times (\delta \underset{\sim}{\phi})$$

$$= -\underset{\sim}{\Lambda} \times \partial_\mu \underset{\sim}{\phi} - \partial_\mu \underset{\sim}{\Lambda} \times \underset{\sim}{\phi} + g(\delta \underset{\sim}{W}_\mu) \times \underset{\sim}{\phi} +$$

$$- g \underset{\sim}{W}_\mu \times (\underset{\sim}{\Lambda} \times \underset{\sim}{\phi})$$

$$= \underline{-\underset{\sim}{\Lambda} \times \partial_\mu \underset{\sim}{\phi}} - \partial_\mu \underset{\sim}{\Lambda} \times \underset{\sim}{\phi} + g(\delta \underset{\sim}{W}_\mu) \times \underset{\sim}{\phi} +$$

$$- g(\underset{\sim}{W}_\mu \times \underset{\sim}{\Lambda}) \times \underset{\sim}{\phi} \underline{- g \underset{\sim}{\Lambda} \times (\underset{\sim}{W}_\mu \times \underset{\sim}{\phi})}$$

$$= -\underset{\sim}{\Lambda} \times (\underset{\sim}{D\phi}) + \left[g \delta \underset{\sim}{W}_\mu - \partial_\mu \underset{\sim}{\Lambda} - g(\underset{\sim}{W}_\mu \times \underset{\sim}{\Lambda})\right] \times \underset{\sim}{\phi}$$

From which we get the sufficient condition

$$\boxed{\delta \underset{\sim}{W}_\mu = -\underset{\sim}{\Lambda} \times \underset{\sim}{W}_\mu + \frac{1}{g} \partial_\mu \underset{\sim}{\Lambda}}$$

We have introduced the field $\underset{\sim}{W}_\mu$ generalizing A_μ. Now we generalize the field strength tensor $F_{\mu\nu}$.

The tensor $\underset{\sim}{W}_{\mu\nu}$ will be a vector under $SO(3)$ and must transform as

$$\delta(\underset{\sim}{W}_{\mu\nu}) = -\underset{\sim}{\Lambda} \times \underset{\sim}{W}_{\mu\nu}$$

First note that

$$\delta\left[\partial_\mu \underset{\sim}{W}_\nu - \partial_\nu \underset{\sim}{W}_\mu\right] =$$

$$= \partial_\mu\left(-\underset{\sim}{\Lambda} \times \underset{\sim}{W}_\nu + \frac{1}{g}\partial_\nu \underset{\sim}{\Lambda}\right) - \partial_\nu\left(-\underset{\sim}{\Lambda} \times \underset{\sim}{W}_\mu + \frac{1}{g}\partial_\mu \underset{\sim}{\Lambda}\right)$$

$$= -\left(\partial_\mu \underset{\sim}{\Lambda} \times \underset{\sim}{W}_\nu - \partial_\nu \underset{\sim}{\Lambda} \times \underset{\sim}{W}_\mu\right) - \Lambda \times \left(\partial_\mu \underset{\sim}{W}_\nu - \partial_\nu \underset{\sim}{W}_\mu\right) +$$

$$+ \frac{1}{g}\left(\partial_\mu \partial_\nu \underset{\sim}{\Lambda} - \partial_\nu \partial_\mu \underset{\sim}{\Lambda}\right).$$

$$= -\Lambda \times \left(\partial_\mu \underset{\sim}{W}_\nu - \partial_\nu \underset{\sim}{W}_\mu\right) - \left(\partial_\mu \underset{\sim}{\Lambda} \times \underset{\sim}{W}_\nu - \partial_\nu \underset{\sim}{\Lambda} \times \underset{\sim}{W}_\mu\right)$$

secondly note that

$$\delta\left[g W_\mu \times W_\nu \right] = g\left[-\Lambda \times W_\mu + \frac{1}{g}\partial_\mu \Lambda \right] \times W_\nu +$$
$$+ g W_\mu \times \left[-\Lambda \times W_\nu + \frac{1}{g}\partial_\nu \Lambda \right]$$

$$= -g \Lambda \times (W_\mu \times W_\nu) + \partial_\mu \Lambda \times W_\nu - \partial_\nu \Lambda \times W_\mu$$

So we define

$$\overline{W}_{\mu\nu} \equiv \partial_\mu \overline{W}_\nu - \partial_\nu \overline{W}_\mu + g \overline{W}_\mu \times \overline{W}_\nu$$

and we have

$$\delta \overline{W}_{\mu\nu} = - \overline{\Lambda} \times \overline{W}_{\mu\nu}$$

And so

$$\overline{W}_{\mu\nu} \cdot \overline{W}^{\mu\nu}$$

is a scalar under SO(3). Finally we write the Lagrangian density as

$$\mathcal{L} = \frac{1}{2} \left(D_\mu \bar{\phi} \right) \cdot \left(D^\mu \bar{\phi} \right) - \frac{m^2}{2} \bar{\phi} \cdot \bar{\phi} +$$
$$- \frac{1}{4} \overline{W}_{\mu\nu} \cdot \overline{W}_{\mu\nu}$$

The total gauge invariant Lagrangian reads

$$\mathcal{L} = \frac{1}{2}(D_\mu \bar{\Phi})\cdot(D^\mu \bar{\phi}) - \frac{m^2}{2}\bar{\Phi}\cdot\bar{\Phi} - \frac{1}{4}\bar{W}_{\mu\nu}\cdot\bar{W}^{\mu\nu}$$

with $$\bar{W}_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + g W_\mu \times W_\nu$$

and $$D_\mu \bar{\Phi} = \partial_\mu \bar{\Phi} + g W_\mu \times \bar{\Phi}$$

The eqs of motion can be obtained from

$$\frac{\partial \mathcal{L}}{\partial(W^i_\alpha)} = (D_\mu \bar{\Phi})^j \frac{\partial (g W_\mu \times \bar{\Phi})^j}{\partial(W^i_\alpha)} - \frac{1}{2}\bar{W}^j_{\mu\nu}\frac{\partial \bar{W}^{\mu\nu j}}{\partial W^i_\alpha}$$

$$= (D_\mu \bar{\Phi})^j \frac{\partial}{\partial(W^i_\alpha)} g\epsilon^{jkl} W^k_\mu \bar{\Phi}^l - \frac{1}{2}\bar{W}^j_{\mu\nu}\frac{\partial g\epsilon^{jkl} W^{\mu k} W^{\nu l}}{\partial W^i_\alpha}$$

$$= (D_\alpha \bar{\Phi})^j g\epsilon^{jil}\bar{\Phi}^l - \bar{W}^j_{\alpha\nu} g\epsilon^{jil} W^{\nu l}$$

$$= -g\epsilon^{ijl}(D_\alpha\bar{\Phi})^j\bar{\phi}^l + g\epsilon^{ijl}\bar{W}^j_{\alpha\nu}W^{\nu l}$$

$$= -g\left[(D_\alpha\bar{\Phi})\times\bar{\Phi}\right]_i + g\left[\bar{W}_{\alpha\nu}\times W^\nu\right]_i$$