Lagrangian Formulation - Symmetries and Gauge Fields

3.1. Lagrangian Formulation of Particle Mechanics

Newton's second law

$$m\frac{d^2x}{dt^2} = -\frac{dV}{dx}$$

can be recovery from a variational principle.

By writing

$$L = T - V = \frac{m\dot{x}^2}{2} - V(x)$$

and

$$S = \int_{t_1}^{t_2} L(x,\dot{x})\,dt$$

we get

$$\delta S = 0 \implies \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\left[m\dot{x}\right] + \frac{\partial V}{\partial x} = 0$$

$$m\ddot{x} = -\frac{dV}{dx}$$

3.2 The real scalar field: variational principle and Noether's theorem

We move from a point particle at position

$$x(t)$$

to a field

$$\phi(x^\mu) = \phi(x, y, z, t)$$

obeying a field equation. For instance the Klein-Gordon equation

$$(\Box + m^2)\phi = 0$$

Oskar Klein (1894 - 1977) - Swedish
Walter Gordon (1893 - 1939) - German

This should be interpreted as a field equation and not a single-particle equation. Particularly the meaning of m will become clear only after quantization.

From classical particle mechanics

$$L = L(q_i, \dot{q}_i) \qquad i = 1, \dots, n$$

$$S = \int dt\, L(q_i, \dot{q}_i)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

to classical field theory

$$L = \int d^3x\, \mathcal{L}\left(\phi(\vec{x},t), \phi'(\vec{x},t), \dot{\phi}(\vec{x},t)\right)$$

$$S = \int dt\, L = \int dt\, d^3x\, \mathcal{L}$$

$$= \int \mathcal{L}(\phi, \partial_\mu \phi)\, d^4x$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

Example: Klein-Gordon

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \partial^\mu \phi + m^2 \phi$$

$$\therefore \quad \Box \phi + m^2 \phi = 0$$

$$\boxed{(\Box + m^2)\phi = 0}$$

Variation of the Action

Let us consider an action of the form

$$S = \int \mathcal{L}(\phi, \partial_\mu \phi, x^\mu)\, d^4x$$

We shall consider variations of this action with two goals :

i) deduce the Euler-Lagrange equations

ii) deduce the Noether theorem

We perform the following variations

$$\begin{cases} x^\mu \longrightarrow x'^\mu = x^\mu + \delta x^\mu \\ \phi(x) \longrightarrow \phi'(x) = \phi(x) + \delta\phi(x) \end{cases}$$

$\delta\phi \rightarrow$ functional variation on ϕ

We may define the total variation on the field, due to the functional variation an to the x^μ variation as

$$\Delta\phi(x) = \phi'(x') - \phi(x)$$

then we may write

$$\Delta\phi(x) = \phi'(x') - \phi(x') + \phi(x') - \phi(x)$$

$$= \delta\phi + (\partial_\mu \phi)\,\delta x^\mu$$

The variation in the action reads

$$\delta S = \int \mathcal{L}(\phi', \partial_\mu \phi', x'^\mu)\, d^4x'$$

$$- \int \mathcal{L}(\phi, \partial_\mu \phi, x^\mu)\, d^4x$$

We relate d^4x' to d^4x by the Jacobian of the transf. $x^\mu \to x'^\mu$

$$d^4x' = \det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right) d^4x$$

$$x'^\mu = x^\mu + \delta x^\mu$$

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu + \partial_\nu(\delta x^\mu)$$

In matricial form we may write

$$\left(\frac{\partial x'^\mu}{\partial x^\nu} \right) = I + \epsilon$$

where I is the identity and ϵ an infinitesimal matrix. Then we have

$$\cancel{\det(I+\epsilon)}$$

$$\det(I+\epsilon) = e^{\ln \det(I+\epsilon)}$$

$$= e^{\operatorname{tr} \ln (I+\epsilon)}$$

$$\cong e^{\operatorname{tr} \epsilon}$$

$$\cong 1 + \operatorname{tr} \epsilon$$

thus

$$\left| \frac{\partial x'^\mu}{\partial x^\nu} \right| = 1 + \partial_\mu (\delta x^\mu)$$

The variation in the action then reads

$$\delta S = \int \mathcal{L}(\phi', \partial_\mu \phi', x'^\mu)\, d^4x' - \int \mathcal{L}(\phi, \partial_\mu \phi, x^\mu)\, d^4x$$

$$= \int \left\{ \underbrace{\left[\mathcal{L}(\phi', \partial_\mu \phi', x'^\mu) - \mathcal{L}(\phi, \partial_\mu \phi, x^\mu) \right]}_{\delta \mathcal{L}} + \mathcal{L}\, \partial_\mu (\delta x^\mu) \right\} d^4x$$

$$= \int \left[\delta \mathcal{L} + \mathcal{L}\, \partial_\mu (\delta x^\mu) \right] d^4x$$

with

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta (\partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial x^\mu} \delta x^\mu$$

regrouping the last two terms (Leibniz rule)

$$\delta S = \int \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta (\partial_\mu \phi) + \partial_\mu (\mathcal{L}\, \delta x^\mu) \right] d^4x$$

$$\delta S = \int \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] \delta\phi \, d^4x \; +$$

$$+ \int \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi + \mathcal{L} \, \delta x^\mu \right] d^4x$$

Considering the region of integration R and using Gauss divergence theorem we write

$$\delta S = \int_R \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] \delta\phi \, d^4x \; +$$

$$+ \int_{\partial R} \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi + \mathcal{L} \, \delta x^\mu \right] d\sigma_\mu$$

Assuming $\delta\phi$ and δx^μ to be arbitrary variations which vanish on the boundary ∂R, we get EL equations

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0}$$

On the other hand, assuming EL equations to hold and performing an arbitrary variation on the fields and x^μ not necessarily vanishing on the boundary we have

$$\delta S = \int_{\partial R} \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi + \mathcal{L}\, \delta x^\mu \right] d\sigma_\mu$$

$$= \int \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi + \mathcal{L}\, \delta x^\mu \right] d^4x$$

In this case, if for some specific form of the variations $\delta\phi$ and δx^μ it happens that the action remains invariant, namely $\delta S = 0$, then we have

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi + \mathcal{L}\, \delta x^\mu \right] = 0$$

By using

$$\delta\phi = \Delta\phi - (\partial_\mu \phi)\,\delta x^\mu$$

we rewrite

$$0 = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \left(\Delta\phi - \partial_\nu \phi\, \delta x^\nu\right) + \mathcal{L}\, \delta x^\mu \right]$$

$$= \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - \underbrace{\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \right)}_{\Theta^\mu_{\ \nu}} \delta x^\nu \right]$$

Defining the energy-momentum tensor

$$\Theta^\mu_{\ \nu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L}$$

we get

$$0 = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - \Theta^\mu_{\ \nu} \delta x^\nu \right]$$

Now we focus on the variations $\delta\phi$ and δx^μ for which ~~the~~ S remains invariant.

Assume we have

$$\delta x^\mu = X^\mu{}_\alpha \, \delta\omega^\alpha$$

$$\Delta\phi = \Phi_\alpha \, \delta\omega^\alpha$$

where $\delta\omega^\alpha$ are infinitesimal parameters, and these inf. transfs belong to a general group of symmetries of S. Then we have

$$0 = \partial_\mu \underbrace{\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Phi_\alpha - \Theta^\mu{}_\nu X^\nu{}_\alpha \right]}_{J^\mu{}_\alpha} \delta\omega^\alpha$$

We define the Noether currents

$$J^\mu{}_\alpha = \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Phi_\alpha - \Theta^\mu{}_\nu X^\nu{}_\alpha \right]$$

For transformations leading to a symmetry of the action, $\delta S = 0$, the Noether currents are conserved in the sense

$$\partial_\mu J^\mu{}_\alpha = 0$$

By applying Gauss divergence theorem ~~we~~ ~~have~~ on

$$\int_R (\partial_\mu J^\mu_\alpha)\, d^4x = 0$$

for an arbitrary integration region R, we have

$$\int_{\partial R} J^\mu_\alpha \, d\sigma_\mu = 0$$

By choosing a hyp. of integration with t = const we define

$$Q_\alpha = \int_V J^0{}_\alpha \, d^3x$$

Now, from $\partial_\mu J^\mu_\alpha = 0$ we have

$$\partial_0 J^0_\alpha + \partial_i J^i_\alpha = 0$$

$$\int_V \partial_0 J^0_\alpha \, d^3x + \int_V \partial_i J^i_\alpha \, d^3x = 0$$

$$\frac{d}{dt} \underbrace{\int_V J^0_\alpha d^3x}_{Q_\alpha} + \underbrace{\int_{\partial V} \vec{J} \cdot \hat{n} \, dS}_{0} = 0$$

$$\frac{dQ_\alpha}{dt} = 0$$

and Q_α is a conserved charge

As an example, let us consider translations in space and time

$$\Delta x^\mu = \varepsilon^\mu \,, \qquad \Delta \phi = 0$$

In this case we have

$$X^\mu{}_\nu = \delta^\mu_\nu \,, \qquad \Phi_\mu = 0$$

and the conserved currents read

$$J^\mu{}_\nu = \cancel{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Phi_\nu}^{\,0} - \theta^\mu{}_\lambda \underbrace{X^\lambda{}_\nu}_{\delta^\mu_\nu}$$

$$\boxed{J^\mu{}_\nu = -\theta^\mu{}_\nu}$$

the corresponding conserved charges are then

$$Q_\nu = \int_V \theta^0{}_\nu \, d^3x$$

For $\nu = 0$, we have

$$Q_0 = \int_V \Theta_0^0 \, d^3x$$

$$= \int \left[\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial_0 \phi - \delta_0^0 \mathcal{L} \right] d^3x$$

$$= \int \left[\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L} \right] d^3x$$

$$= H$$

where $H = \int \mathcal{H} \, d^3x$

with

$$\mathcal{H} \equiv \dot{\phi}\pi - \mathcal{L}$$

being the Hamiltonian density and

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

the canonical momentum conjugated to ϕ.

Thus $Q_0 = H = \int \mathcal{H}\, d^3x$ is the energy of the field configuration.

Similarly,

$$Q_i = \int \Theta^0{}_i\, d^3x = \int \left[\frac{\partial \mathcal{L}}{\partial \dot\phi} \partial_i \phi - \delta^0{}_i \mathcal{L} \right] d^3x$$

$$= \int \frac{\partial \mathcal{L}}{\partial \dot\phi} \partial_i \phi \; d^3x$$

$$= \int \pi\, \partial_i \phi\, d^3x$$

is the linear momentum of the field configuration.

For the specific case of the Klein-Gordon Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2} \phi^2 ,$$

we have

$$\Theta^{\mu\nu} = g^{\mu\rho}\, \Theta_\rho{}^\nu = \Theta^\mu{}_\rho\, g^{\rho\nu}$$

$$\Theta^{\mu\nu} = \Theta^{\mu}{}_{\rho}\, g^{\rho\nu}$$

$$= \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\rho \phi - \delta^\mu_\rho \mathcal{L} \right] g^{\rho\nu}$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} \mathcal{L}$$

$$= (\partial^\mu \phi)(\partial^\nu \phi) - g^{\mu\nu} \mathcal{L}$$

which is symmetric in μ, ν.

We have seen, the energy-momentum tensor $\Theta^{\mu}{}_{\nu}$ is defined as

$$\Theta^{\mu}{}_{\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^{\mu}_{\nu} \mathcal{L}$$

or

$$\cancel{\Theta^{\mu\nu} = (\partial^\mu \phi)(\partial^\nu \phi) - g^{\mu\nu} \mathcal{L}}$$

$$\Theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} \mathcal{L}$$

In the case of

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2}\phi^2$$

We have calculated

$$\Theta^{\mu\nu} = \partial^\mu \phi \, \partial^\nu \phi - \frac{1}{2} g^{\mu\nu}(\partial_\lambda \phi)(\partial^\lambda \phi) + \frac{m^2 g^{\mu\nu} \phi^2}{2}$$

$$= \partial^\mu \phi \, \partial^\nu \phi - g^{\mu\nu} \mathcal{L}$$

which is symmetric in μ, ν.

Note that for space-time translation

$$\Delta x^\mu = \varepsilon^\mu$$

we get

$$\partial_\mu J^\mu{}_\nu = 0$$

with $J^\mu{}_\nu = -\Theta^\mu{}_\nu$

that means

$$\partial_\mu \Theta^{\mu\nu} = 0$$

However, if we add $\partial_\lambda f^{\lambda\mu\nu}$ with

$$f^{\lambda\mu\nu} = -f^{\mu\lambda\nu}$$

we still have

$$\partial_\mu \left[\Theta^{\mu\nu} + \partial_\lambda f^{\lambda\mu\nu}\right] = \partial_\mu \Theta^{\mu\nu} + \partial_\mu \partial_\lambda f^{\lambda\mu\nu}$$

$$= 0$$

then we define

$$T^{\mu\nu} \equiv \Theta^{\mu\nu} + \partial_\lambda f^{\lambda\mu\nu}$$

where $f^{\lambda\mu\nu}$, antisymmetric in λ, μ, is chosen such that $T^{\mu\nu}$ is symmetric.

Namely, we demand

$$T^{\mu\nu} = T^{\nu\mu}$$

$$\Theta^{\mu\nu} + \partial_\lambda f^{\lambda\mu\nu} = \Theta^{\nu\mu} + \partial_\lambda f^{\lambda\nu\mu}$$

$$\partial_\lambda \left[f^{\lambda\mu\nu} - f^{\lambda\nu\mu} \right] = \Theta^{\nu\mu} - \Theta^{\mu\nu}$$

Note that

$$\cancel{\int \partial_\lambda f^{\lambda 0 \nu} d^3x + \int \partial_\lambda f}$$

$$\int \partial_\lambda f^{\lambda 0 \nu} d^3x = \overset{0}{\cancel{\int \partial_0 f^{00\nu} d^3x}} + \int \partial_i f^{i0\nu} d^3x$$

$$= \int \partial_i f^{i0\nu} d^3x$$

$$= \int_\sigma f^{i0\nu} d\sigma_i = 0$$

and thus

$$P_\nu = \int_V \Theta^0{}_\nu \, d^3x$$

$$= \int_V \left(\Theta^0{}_\nu + \partial_\lambda f^{\lambda 0}{}_\nu \right) d^3x$$

$$= \int_V T^0{}_\nu \, d^3x$$

$$P^{\nu} = \int_V \Theta^{0\nu} \, d^3x$$

$$= \int_V \left(\Theta^{0\nu} + \partial_\lambda f^{\lambda 0 \nu} \right) d^3x$$

$$= \int_V T^{0\nu} \, d^3x$$

Recall that Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = - \frac{8\pi G}{c^2} T_{\mu\nu}$$

demand $T_{\mu\nu}$ to be symmetric.

For spatial rotations, we have

$$x^i \longrightarrow x^{i'} = \left[e^{i\frac{\theta_k L_k}{\hbar}} \right]^i{}_j \, x^j$$

For the inf. case:

$$x^{i'} = \left[\mathbb{1} + \frac{i\theta^k L_k}{\hbar} \right]^i{}_j \, x^j$$

$$= \left[\delta^i{}_j + \frac{i\theta^k}{\hbar} (L_k)^i{}_j \right] x^j$$

$$= \left[\delta^i{}_j + \frac{i\theta^k}{\hbar} \left(-i \epsilon^i{}_{jk} \hbar \right) \right] x^j$$

$$= \left[\delta^i{}_j + \underbrace{\epsilon^i{}_{jk} \theta^k}_{\omega^i{}_j} \right] x^j$$

By defining $\omega^i{}_j \equiv \epsilon^i{}_{jk} \theta^k$ we see

$$x^{i'} = x^i + \omega^i{}_j x^j \quad \text{with} \quad \omega^i{}_j = -\omega_j{}^i$$

In terms of a general Lorentz transformation, we have

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

where Λ satisfies $\Lambda^T G \Lambda = G$ or equivalently, $\Lambda G \Lambda^T = G$. In components, they read

$$(\Lambda^T)_{\mu}{}^{\alpha} (G)_{\alpha\beta} (\Lambda)^{\beta}{}_{\nu} = (G)_{\mu\nu}$$

$$(\Lambda^T)_{\mu}{}^{\alpha} g_{\alpha\beta} \Lambda^{\beta}{}_{\nu} = g_{\mu\nu}$$

$$\boxed{\Lambda^{\alpha}{}_{\mu} g_{\alpha\beta} \Lambda^{\beta}{}_{\nu} = g_{\mu\nu}}$$

or

$$\Lambda_{\mu}{}^{\alpha} g_{\alpha\beta} (\Lambda^T)^{\beta}{}_{\nu} = g_{\mu\nu}$$

$$\boxed{\Lambda_{\mu}{}^{\alpha} g_{\alpha\beta} \Lambda_{\nu}{}^{\beta} = g_{\mu\nu}}$$

By considering infinitesimal transfs, we get

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \epsilon^\mu{}_\nu$$

$$\Lambda^\alpha{}_\mu \, g_{\alpha\rho} \, \Lambda^\rho{}_\nu = g_{\mu\nu}$$

$$(\delta^\alpha_\mu + \epsilon^\alpha{}_\mu) \, g_{\alpha\rho} \, (\delta^\rho_\nu + \epsilon^\rho{}_\nu) = g_{\mu\nu}$$

$$g_{\mu\nu} + \epsilon_{\mu\nu} + \epsilon_{\nu\mu} = g_{\mu\nu}$$

$$\therefore \boxed{\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}}$$

$$\boxed{\delta x^\mu = \epsilon^\mu{}_\nu \, x^\nu \quad \text{with} \quad \epsilon_{\mu\nu} = -\epsilon_{\nu\mu}}$$

Now we rewrite

$$\delta x^\mu = X^\mu{}_{\alpha\rho} \, \epsilon^{\alpha\rho}$$

$$= \frac{1}{2}\left[-x_\alpha \delta^\mu_\rho + x_\rho \delta^\mu_\alpha\right] \epsilon^{\alpha\rho}$$

Summary

Recall we had defined the energy-momentum tensor as

$$\Theta^{\mu}{}_{\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^{\mu}{}_{\nu} \mathcal{L}$$

or simply

$$\Theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}$$

In the case of

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{m^2}{2} \phi^2$$

we have calculated

$$\Theta^{\mu\nu} = \partial^\mu \phi \, \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L}$$

which is symmetric in $\mu\nu$.

We have seen we may define

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda f^{\lambda\mu\nu}$$

with $f^{\lambda\mu\nu}$ antisymmetric in λ, μ so that

$T^{\mu\nu}$ is conserved and symmetric in $\mu\nu$

We have seen that translational symmetry

$$\delta x^\mu = \epsilon^\mu$$

lead to the four conservation laws

$$\partial_\mu \Theta^{\mu\nu} = 0$$

or equivalently

$$\boxed{\partial_\mu T^{\mu\nu} = 0}$$

We consider thus infinitesimal Lorentz transformations

$$\delta x^\mu = \epsilon^\mu{}_\nu \, x^\nu$$

with $\epsilon_{\mu\nu} = - \epsilon_{\nu\mu}$. We have $\frac{4^2-4}{2} = 6$ independent parameters. We may rewrite δx^μ as

$$\delta x^\mu = X^\mu{}_{\alpha\beta} \, \epsilon^{\alpha\beta} \, ,$$

and then

$$\begin{aligned} X^\mu{}_{\alpha\beta} \, \epsilon^{\alpha\beta} &= \epsilon^\mu{}_\nu \, x^\nu \\ &= \delta^\mu_\alpha \, \epsilon^\alpha{}_\nu \, x^\nu \\ &= \delta^\mu{}_\alpha \, \epsilon^{\alpha\beta} \, g_{\beta\nu} \, x^\nu \\ &= \delta^\mu_\alpha \, g_{\beta\nu} \, x^\nu \, \epsilon^{\alpha\beta} \\ &= \delta^\mu{}_\alpha \, x_\beta \, \epsilon^{\alpha\beta} \\ &= \frac{1}{2} \left(\delta^\mu{}_\alpha \, x_\beta - \delta^\mu{}_\beta \, x_\alpha \right) \epsilon^{\alpha\beta} \end{aligned}$$

comparing with the general transf. equations

$$\Delta x^\mu = X^\mu{}_\nu \, \delta \omega^\nu$$

$$\Delta \phi = \Phi_\mu \, \delta \omega^\mu$$

We put $\Phi_\mu = 0$ in the second and, in the first, we consider ν as a collective index $\alpha\beta$.

Recall, the Noether currents are

$$J^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Phi_\nu - \theta^\mu{}_\kappa X^\kappa{}_\nu$$

So in the present case we have

$$J^\mu{}_{\alpha\beta} = -\theta^\mu{}_\nu \, X^\nu{}_{\alpha\beta}$$

Since $\theta^{\mu\nu}$ is already symmetric, we put $T^{\mu\nu} = \theta^{\mu\nu}$ and have

$$J^\mu{}_{\alpha\beta} = - T^\mu{}_\nu \, X^\nu{}_{\alpha\beta}$$

or simply

$$J^{\mu\alpha\beta} = - T^{\mu}{}_{\nu} X^{\nu\alpha\beta}$$

Substituting

$$X^{\nu\alpha\beta} = \frac{1}{2}\left(\eta^{\nu\alpha} x^{\beta} - \eta^{\nu\beta} x^{\alpha} \right)$$

we get

$$J^{\mu\alpha\beta} = -\frac{1}{2}\left(\eta^{\nu\alpha} x^{\beta} - \eta^{\nu\beta} x^{\alpha}\right) T^{\mu}{}_{\nu}$$

$$= -\frac{1}{2}\left(T^{\mu\alpha} x^{\beta} - T^{\mu\beta} x^{\alpha} \right)$$

Note that

$$\partial_{\mu} J^{\mu\alpha\beta} = -\frac{1}{2}\Big[(\partial_{\mu} T^{\mu\alpha}) x^{\beta} + T^{\mu\alpha} \delta^{\beta}_{\mu} + - (\partial_{\mu} T^{\mu\beta}) x^{\alpha} - T^{\mu\beta} \delta^{\alpha}_{\mu} \Big]$$

$$= -\frac{1}{2}\left[T^{\beta\alpha} - T^{\alpha\beta} \right] = 0$$

The $\mu = 0$ component of $J^{\mu\alpha\beta}$ is proportional to the ang. momentum of the field, which is defined by

$$M^{\mu\nu} = \int \left(T^{0\mu} x^{\nu} - T^{0\nu} x^{\mu} \right) d^3x$$

that is

$$M^{\mu\nu} = \int \mathcal{M}^{0\mu\nu} \, d^3x$$

with

$$\mathcal{M}^{0\mu\nu} = T^{0\mu} x^{\nu} - T^{0\nu} x^{\mu}$$

We have $M^{\mu\nu}$ as a conserved quantity

$$\frac{d}{dt} M^{\mu\nu} = 0$$

while

$$\partial_{\alpha} \mathcal{M}^{\alpha\mu\nu} = 0$$

Note that the time conservation of $M^{\mu\nu}$

$$\frac{d}{dt} M^{\mu\nu} = 0$$

opens into the conservation of six quantities, namely

$$M^{12}, \quad M^{23}, \quad M^{31}$$

and

$$M^{01}, \quad M^{02}, \quad M^{03}$$

In the following, we turn our attention to electric charge conservation

The complex scalar field

Now ~~w~~ we have two real scalar fields ϕ_1 and ϕ_2.

Define

$$\phi = \frac{\sqrt{2}}{2}(\phi_1 + i\phi_2)$$

$$\phi^* = \frac{\sqrt{2}}{2}(\phi_1 - i\phi_2)$$

and write the action as

$$\mathcal{L} = \partial_\mu \phi \, \partial^\mu \phi^* - m^2 \phi^* \phi$$

The E.L. equations read

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \partial^\mu \phi^* + m^2 \phi^*$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \right) - \frac{\partial \mathcal{L}}{\partial \phi^*} = \partial_\mu \partial^\mu \phi + m^2 \phi$$

that is

$$(\Box + m^2)\phi = 0$$

$$(\Box + m^2)\phi^* = 0$$

The Lagr. is invariant under

$$\phi \to e^{-i\Lambda}\phi \qquad\qquad \Lambda \in \mathbb{R}$$

$$\phi^* \to e^{i\Lambda}\phi^*$$

This is a gauge transf. of the first kind.

<u>Group</u>: $U(1) \cong SO(2)$

$0 \leq \Lambda < 2\pi$ $\qquad g = e^{-i\Lambda}$

or $\qquad g = \begin{pmatrix} \cos\Lambda & -\sin\Lambda \\ \sin\Lambda & \cos\Lambda \end{pmatrix}$

<u>vector field</u>: $\mathcal{F}$ complex functions defined on M^4

$$\phi : M^4 \longrightarrow \mathbb{C}$$

$$\phi \in \mathcal{F}$$

or : vector field : $\Phi : M^4 \longrightarrow \mathbb{R}^2$

$$x^\mu \longmapsto \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}$$

The finite transf.

$$\phi \to e^{-i\Lambda}\phi$$

$$\phi^* \to e^{i\Lambda}\phi^*$$

leads to the infinitesimal transformations

$$\delta\phi = -i\Lambda\phi$$

$$\delta\phi^* = i\Lambda\phi$$

And since Λ does not depend on space-time,

we also have

$$\delta(\partial_\mu\phi) = -i\Lambda\partial_\mu\phi$$

$$\delta(\partial_\mu\phi^*) = i\Lambda\partial_\mu\phi^*$$

Again considering the Noether's theorem notation

$$\Delta x^\mu = X^\mu{}_\nu\,\delta\omega^\nu, \quad \Delta\phi = \Phi_\mu\,\delta\omega^\mu$$

We have

$$\begin{cases} X = 0 \\ \Phi = -i\phi \\ \Phi^* = i\phi \end{cases}$$

Recall Noether's Theorem

$$\begin{cases} \delta x^\mu = X^\mu{}_\alpha \, \delta\omega^\alpha \\ \Delta\phi^a = \Phi^a{}_\alpha \, \delta\omega^\alpha \end{cases}$$

conserved current

$$J^\mu{}_\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \Phi^a{}_\alpha - \theta^\mu{}_\nu X^\nu{}_\alpha$$

In the present case, we have one infinitesimal parameter Λ, no space-time transf. $(X^\mu \equiv 0)$, and two fields ϕ, ϕ^* $(a = 1, 2)$. Thus there is one conserved current which reads

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \Phi^*$$

The symmetry coefficients ar

$$\Phi = -i\phi \quad \text{and} \quad \Phi^* = i\phi$$

substituting in the current J^μ we get

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(-i\phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}(i\phi^*)$$

$$= -i\,\partial^\mu\phi^*\,\phi + i\,\partial^\mu\phi\,\phi^*$$

$$= i\left(\phi^*\partial^\mu\phi - \phi\,\partial^\mu\phi^*\right)$$

The time conserved charge is

$$Q = \int J^0\, dV$$

$$Q = i\int \left[\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right] dV$$

Note that, by virtue of the equations of motion, we have explicitly

$$\partial_\mu J^\mu = i\left(\phi^*\Box\phi - \phi\Box\phi^*\right)$$

$$= i\left(\phi^*(-m^2\phi) - \phi(-m^2\phi^*)\right) = 0$$