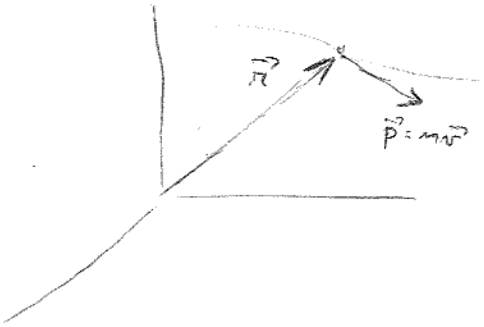
Angular Momentum of a One-Particle System

Nov 29th
2013

Classical Approach



$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$$

linear momentum

$$\vec{p} \equiv m\vec{v}$$

angular momentum:

$$\vec{L} = \vec{r} \times \vec{p}$$

in components, $\vec{L} = L_x\hat{i} + L_y\hat{j} + L_z\hat{k}$

with

$$L_x = y p_z - z p_y$$

$$L_y = z p_x - x p_z$$

$$L_z = x p_y - y p_x$$

Still in classical mechanics,

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

68 Quantum Mechanical Approach

The classical quantities L_x, L_y, L_z will be sent to Hermitian operators $\hat{L}_x, \hat{L}_y, \hat{L}_z$. By considering the quantization rule

$$x \to x \qquad y \to y \qquad z \to z$$

$$p_x \to -i\hbar \frac{\partial}{\partial x} \qquad p_y \to -i\hbar \frac{\partial}{\partial y} \qquad p_z \to -i\hbar \frac{\partial}{\partial z}$$

we write

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

In order to know which states we can prepare, possibly being simultaneous eigenvectors of these operators we calculate their commutators

$$[L_x, L_y] = (-i\hbar)^2 \left[y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} , z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right]$$

$$= (-i\hbar)^2 \Big\{ y\frac{\partial}{\partial x} + \cancel{yz\frac{\partial^2}{\partial z \partial x}} - xy\frac{\partial^2}{\partial z^2} - \cancel{z^2\frac{\partial^2}{\partial x \partial y}} + xz\frac{\partial^2}{\partial y \partial z} +$$

$$- \cancel{zy\frac{\partial^2}{\partial x \partial z}} + z^2\frac{\partial^2}{\partial x \partial y} + xy\frac{\partial^2}{\partial z^2} - x\frac{\partial}{\partial y} - xz\frac{\partial^2}{\partial y \partial z} \Big\}$$

$$= (-i\hbar)\underbrace{(-i\hbar)\Big\{ y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \Big\}}_{-\hat{L}_z}$$

$$\therefore \quad [\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$$

similarly we have

$$[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x$$

$$[\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y .$$

So we see $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$ do not commute among themselves and we cannot specify simultaneously their eigenvalues.

However, we have another important Hermitian operator, namely

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 .$$

Note that

$$[\hat{L}^2, \hat{L}_x] = [L_x^2 + L_y^2 + L_z^2 , L_x]$$

$$= [L_y^2 + L_z^2 ; L_x]$$

$$= L_y [L_y, L_x] + \overbrace{[L_y, L_x]}^{-i\hbar L_z} L_y +$$

$$+ L_z [L_z, L_x] + \underbrace{[L_z, L_x]}_{i\hbar L_y} L_z$$

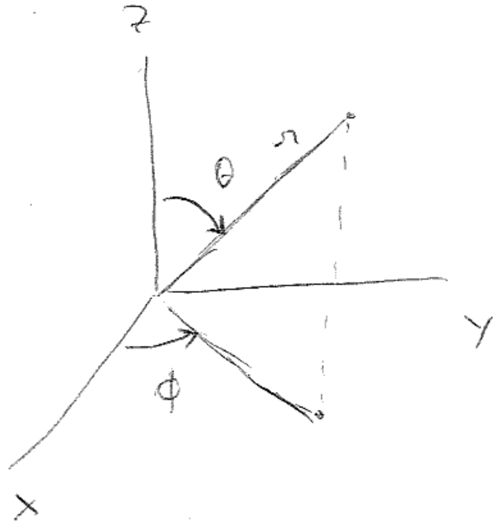
$$= -i\hbar \{ L_y L_z + L_z L_y \} +$$

$$+ i\hbar \{ L_z L_y + L_y L_z \} = 0 \quad //$$

and similarly

$$[\hat{L}^2, \hat{L}_y] = [\hat{L}^2, \hat{L}_z] = 0$$

So we see we may choose $\hat{L}^2$ and one of the components $\hat{L}_x$, $\hat{L}_y$ or $\hat{L}_z$. Conventionally we choose $\hat{L}^2$ and $\hat{L}_z$. Since we are going to study systems with spherical symmetry (For instance the hydrogen atom), it is easier to work with spherical coordinates



$$(x, y, z) \longrightarrow (r, \theta, \phi)$$

$$\begin{cases} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r \cos\theta \end{cases}$$

In spherical coordinates r, θ, ϕ, we may rewrite the angular momentum operators as

$$\begin{cases} \hat{L}_x = i\hbar \left(\sin\phi \dfrac{\partial}{\partial\theta} + \cot\theta \cos\phi \dfrac{\partial}{\partial\phi} \right) \\ \hat{L}_y = -i\hbar \left(\cos\phi \dfrac{\partial}{\partial\theta} - \cot\theta \sin\phi \dfrac{\partial}{\partial\phi} \right) \\ \hat{L}_z = -i\hbar \dfrac{\partial}{\partial\phi} \end{cases}$$

and

$$\hat{L}^2 = -\hbar^2 \left(\frac{\partial^2}{\partial \theta^2} + \cot\theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right)$$

We have chosen $\hat{L}^2$ and $\hat{L}_z$ to work with. Since $\hat{L}^2$ and $\hat{L}_z$ commute, namely,

$$0 = [\hat{L}^2, \hat{L}_z],$$

our task now is to find simultaneously eigenfunctions for $\hat{L}^2$ and $\hat{L}_z$. In spherical coordinates we look for functions $f(r, \theta, \phi)$ such that

$$\begin{cases} \hat{L}_z f = b f \\ \hat{L}^2 f = c f \end{cases}$$

From the fact that the operators $\hat{L}_z$ and $\hat{L}^2$ do not contain r we may write $f(r, \theta, \phi) = R(r) \cdot Y(\theta, \phi)$ and solve the simpler problem

Nov, 29th
2013
Thibes

$$\begin{cases} \hat{L}_z \, Y(\theta,\phi) = b \, Y(\theta,\phi) \\ \hat{L}^2 \, Y(\theta,\phi) = c \, Y(\theta,\phi) \end{cases}$$

with

$$\begin{cases} \hat{L}_z = -i\hbar \dfrac{\partial}{\partial \phi} \\ \hat{L}^2 = -\hbar^2 \left(\dfrac{\partial^2}{\partial \theta^2} + \cot\theta \dfrac{\partial}{\partial \theta} + \dfrac{1}{\sin^2\theta} \dfrac{\partial^2}{\partial \phi^2} \right) \end{cases}$$

We begin with the first equation

$$\hat{L}_z Y(\theta,\phi) = b \, Y(\theta,\phi)$$

which explicitly means

$$-i\hbar \frac{\partial}{\partial \phi} Y(\theta,\phi) = b \, Y(\theta,\phi)$$

Since the differential operator $\hat{L}_z = -i\hbar \dfrac{\partial}{\partial \phi}$ does not contain θ, we write

$$Y(\theta,\phi) \doteq S(\theta) \, T(\phi)$$

and obtain

$$-i\hbar \frac{\partial}{\partial\phi} S(\theta)T(\phi) = \ell\, S(\theta)\,T(\phi)$$

$$-i\hbar\, S(\theta)\frac{dT}{d\phi} = \ell\, S(\theta)\, T$$

$$-i\hbar \frac{dT}{d\phi} = \ell T$$

$$\frac{dT}{T} = \frac{i\ell}{\hbar}\, d\phi$$

$$\therefore \boxed{T(\phi) = A e^{\frac{i\ell}{\hbar}\phi}}$$

Since ϕ represents a physical angle, we must impose periodic conditions

$$T(\phi + 2\pi) = T(\phi)$$

$$\cancel{A}\, e^{\frac{i\ell}{\hbar}(\phi+2\pi)} = \cancel{A}\, e^{\frac{i\ell\phi}{\hbar}}$$

$$e^{\frac{i\ell\phi}{\hbar}}\, e^{\frac{i\ell 2\pi}{\hbar}} = e^{\frac{i\ell\phi}{\hbar}}$$

$$\boxed{e^{\frac{i\ell 2\pi}{\hbar}} = 1}$$

thus we must have

$$\frac{b\,2\pi}{\hbar} = 2\pi m, \qquad m \in \mathbb{Z}$$

leading to

$$b = m\hbar, \qquad m = 0, \pm 1, \pm 2, \ldots$$

Summarizing, we have

$$Y(\theta,\phi) = S(\theta)\, e^{im\phi}$$ (the const A may be absorbed into $S(\theta)$)

since different values of the integer m lead to different functions $Y(\theta,\phi)$ we attach an index m to $Y(\theta,\phi)$. So, again, summarizing:

$$\left\{ \begin{aligned} Y^m(\theta,\phi) &= S(\theta)\, e^{im\phi} \\ \hat{L}_3\, Y^m(\theta,\phi) &= m\hbar\, Y^m(\theta,\phi) \\ \hat{L}^2\, Y^m(\theta,\phi) &= c\, Y^m(\theta,\phi) \end{aligned} \right.$$

We must now calculate $S(\theta)$ and c.

Substituting the first equation

$$Y^m(\theta,\phi) = S(\theta)\, e^{im\phi}$$

into the last equation

$$\hat{L}^2\, Y^m(\theta,\phi) = c\, Y^m(\theta,\phi)$$

and recalling the expression

$$\hat{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\,\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\phi^2}\right)$$

we get

$$-\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\,\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\phi^2}\right) S(\theta)\, e^{im\phi} = c\, S(\theta)\, e^{im\phi}$$

$$-\hbar^2\left(S'' + \cot\theta\, S' - \frac{m^2}{\sin^2\theta}\, S\right) = c\, S$$

$$\boxed{S'' + \cot\theta\, S' + \left(\frac{c}{\hbar^2} - \frac{m^2}{\sin^2\theta}\right) S = 0}$$

This is a second order linear equation on $S(\theta)$, homogeneous but with variable coefficients.

Observe that the equation for $S(\theta)$ is parametrized by the integer m and the real c

$$S'' + \cot\theta\, S' + \left(\frac{c}{\ell_1^2} - \frac{m^2}{\sin^2\theta}\right) S = 0$$

therefore we expect the solutions to depend on m and c. It is possible to prove (Ira Levine) that the real coefficient c is given by

$$c = \ell(\ell+1)\,\ell_1^2 , \qquad \ell = 0, 1, 2, \ldots, \ \ell \geq m$$

The solutions $S_{\ell,m}(\theta)$ are the associated Legendre functions. The first associated Legendre functions read

$\ell = 0$: $\quad S_{0,0} = \frac{1}{2}\sqrt{2}$

$\ell = 1$: $\quad S_{1,0} = \frac{1}{2}\sqrt{6}\cos\theta$

$\quad S_{1,\pm 1} = \frac{1}{2}\sqrt{3}\sin\theta$

$\ell = 2$: $\quad S_{2,0} = \frac{1}{4}\sqrt{10}\,(3\cos^2\theta - 1)$

$\quad S_{2,\pm 1} = \frac{1}{2}\sqrt{15}\sin\theta\cos\theta$

Rem:

To be more precise, if we come back to $T(\phi)$ (p. 74), since $T(\phi)$ is to represent part of a wave function, we could have imposed

$$T(\phi + 2\pi) = \pm T(\phi)$$

and we would have gotten the conditions

$$\begin{cases} l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \\ m_l = 0, \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots, \pm l \end{cases}$$

However, this will be valid only for spin angular momentum. For ordinary orbital angular momentum we still have $l = 0, 1, 2, \ldots$ and $|m_l| \leq l$.

Below we summarize our results for orbital angular momentum

$$\begin{cases} Y_l^m(\theta,\phi) = S_{l,m}(\theta)\, T_m(\phi) = \frac{1}{\sqrt{2\pi}} S_{l,m}(\theta)\, e^{im\phi} \\ \hat{L}^2\, Y_l^m(\theta,\phi) = l(l+1)\hbar^2\, Y_l^m(\theta,\phi) \\ \hat{L}_z\, Y_l^m(\theta,\phi) = m\hbar\, Y_l^m(\theta,\phi) \end{cases}$$

THE HYDROGEN ATOM

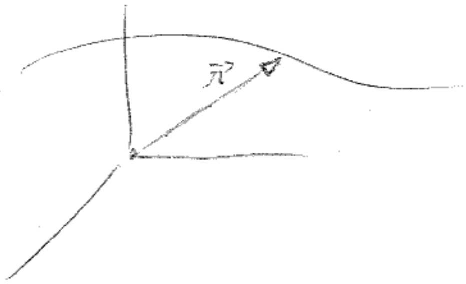
The One-Particle Central-Force Problem

classical analyses: Consider one particle subjected to a conservative central force. A conservative central force can be described by a potential of the form

$$V = V(r)$$

where r is the distance to the fixed center of force. We choose the origin of the coordinate system coinciding with the center of force.

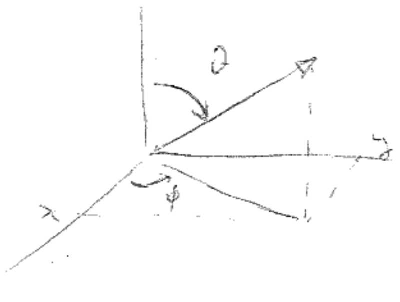
In cartesian coordinates we have



$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r = \sqrt{x^2+y^2+z^2}$$

and in spherical coordinates



$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

The corresponding force is given by

$$\vec{F} = -\vec{\nabla} V = -\frac{\partial V}{\partial x}\hat{\imath} - \frac{\partial V}{\partial y}\hat{\jmath} - \frac{\partial V}{\partial z}\hat{k}$$

In cartesian coordinates $V = V(x, y, z)$ and in spherical coordinates $V = V(r)$. Thus we have

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial V}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial V}{\partial \phi}\frac{\partial \phi}{\partial x} = \frac{dV}{dr}\frac{\partial r}{\partial x}$$

From $r = \sqrt{x^2+y^2+z^2}$ we have

$$\frac{\partial r}{\partial x} = \frac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2} \cdot 2x = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

and thus

$$\frac{\partial V}{\partial x} = \frac{dV}{dr}\cdot\frac{x}{r}, \quad \frac{\partial V}{\partial y} = \frac{dV}{dr}\frac{y}{r}, \quad \frac{\partial V}{\partial z} = \frac{dV}{dr}\frac{z}{r}.$$

Substituting into $\vec{F} = -\vec{\nabla} V$ we get

$$\vec{F} = \left(-\frac{x}{r}\hat{\imath} - \frac{y}{r}\hat{\jmath} - \frac{z}{r}\hat{k}\right)\frac{dV}{dr} = -\frac{(x\hat{\imath}+y\hat{\jmath}+z\hat{k})}{r}\frac{dV}{dr}$$

$$\vec{F} = -\frac{\vec{r}}{r}\frac{dV}{dr}$$ which means $$\boxed{\vec{F} = -\frac{dV}{dr}\hat{r}}$$

Or we could have reached the same result by considering the expression for the gradient in spherical coordinates

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\theta}\frac{\partial}{\partial \theta} + \frac{1}{r\sin\theta}\hat{\phi}\frac{\partial}{\partial \phi}$$

By considering $V = V(r)$, $\frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial \phi} = 0$, we have

$$\vec{F} = -\vec{\nabla} V = -\frac{dV}{dr}\hat{r}$$

We see indeed $V = V(r)$ characterizes a central force. Still in the classical analyses, the classical angular momentum reads

$$\vec{L} = \vec{r} \times \vec{p}$$

and its time derivative goes as

$$\frac{d\vec{L}}{dt} = \overset{\vec{0}}{\cancel{\dot{\vec{r}} \times \vec{p}}} + \vec{r} \times \dot{\vec{p}}$$

$$\frac{d\vec{L}}{dt} = \vec{r} \times \dot{\vec{p}}$$

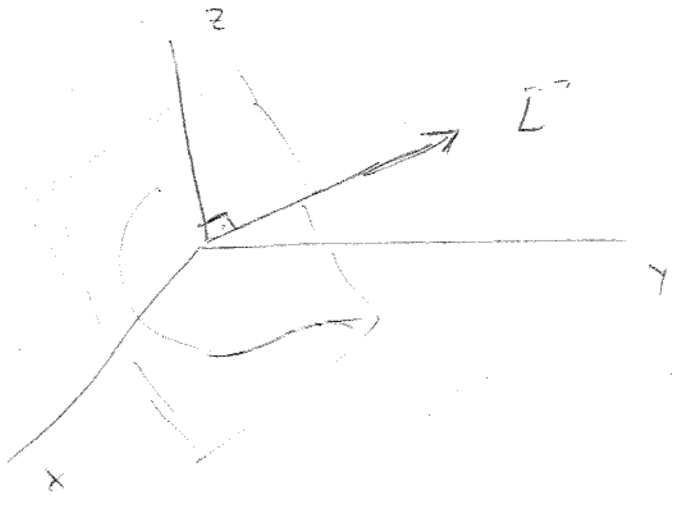
By Newton's second law, $\dot{\vec{p}} = \vec{F}$,

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$$

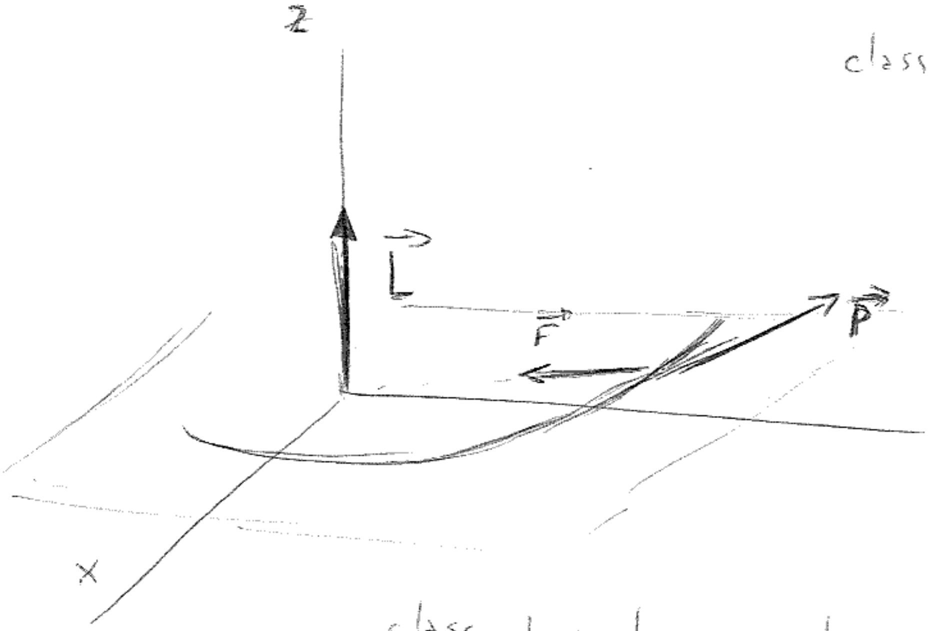
when the force is central, we have thus

$$\frac{d\vec{L}}{dt} = \vec{0} \quad , \quad \underline{\underline{\vec{L} = \text{const}}}$$

Classically, the fact that $\vec{L}$ is constant, implies the trajectory must lie in a plane passing by the center of force.



It is common practice, in classical physics, to choose the Z axis along the constant direction of $\vec{L}$.



classical hamiltonian

$$H(q,p) = \frac{\vec{p}^2}{2m} + V(r)$$

classical mechanics: planar motion in XY plane

<u>quantum analyses</u>:

As usual, in quantum mechanics we perform the transformation

$$\vec{r} \longrightarrow \vec{r}$$
$$\vec{p} \longrightarrow -i\hbar\vec{\nabla}$$

(which is the same as $x \to x$, $y \to y$, $z \to z$, $p_x \to -i\hbar\frac{\partial}{\partial x}$, $p_y \to -i\hbar\frac{\partial}{\partial y}$, $p_z \to -i\hbar\frac{\partial}{\partial z}$)

and write

$$\hat{H} = \hat{T} + \hat{V}$$

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(r)$$

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + V(r)$$

Since we have spherical symmetry, it is easier to work with spherical coordinates.

A direct (but a bit lengthy) calculation* shows that ∇^2 can be written in spherical coordinates as

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2}\cot\theta\frac{\partial}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

By recalling that we had

$$\hat{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial \theta^2} + \cot\theta\frac{\partial}{\partial \theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial \phi^2}\right)$$

we see we can rewrite

$$\boxed{\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{r^2\hbar^2}\hat{L}^2}$$

Therefore, in spherical coordinates, the Hamiltonian can be written as

Dec 2013 Thibis

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(r)$$

$$\boxed{\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) - \frac{1}{2mr^2}\hat{L}^2 + V(r)}$$

Inspired by the fact that in classical mechanics we have angular momentum conservation we calculate

$$[\hat{H}, \hat{L}^2] = [\hat{T}, \hat{L}^2] + [\hat{V}, \hat{L}^2]$$

$$= \left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) - \frac{1}{2mr^2}\hat{L}^2, \hat{L}^2\right] +$$

$$+ [V(r), \hat{L}^2]$$

$$= \left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right), \hat{L}^2\right]$$

$$= 0 \quad //$$

So $\hat{H}$ commutes with $\hat{L}^2$. What about $\hat{L}_z$? Let us check

$$[\hat{H}, \hat{L}_z] = \left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) - \frac{1}{2mr^2}\hat{L}^2 + V(r), \hat{L}_z\right]$$

$$= \left[-\frac{1}{2mr^2}\hat{L}^2, \hat{L}_z \right] = 0$$

Therefore the operators $\hat{H}$, $\hat{L}^2$ and $\hat{L}_z$ mutually commute with each other

$$[\hat{H}, \hat{L}^2] = 0 \qquad [\hat{L}^2, \hat{H}] = 0 \qquad [\hat{L}_z, \hat{H}] = 0$$

$$[\hat{H}, \hat{L}_z] = 0 \qquad [\hat{L}^2, \hat{L}_z] = 0 \qquad [\hat{L}_z, \hat{L}^2] = 0$$

Recall theorem 5:

<u>Theorem #5</u>

If the Hermitian operators $\hat{A}$ and $\hat{B}$ commute, we can select a common complete set of eigenfunctions for them.

It is possible to extend this theorem for the case of N mutually commuting operators. In the present case we have 3 mutually commuting operators $\hat{H}$, $\hat{L}^2$ and $\hat{L}_z$.

We therefore consider a complete set of common eigenfunction ψ for the operators $\hat{H}$, $\hat{L}^2$ and L_z

$$\begin{cases} \hat{H}\psi = E\psi \\ \hat{L}^2\psi = \ell(\ell+1)\hbar^2\psi \\ \hat{L}_z\psi = m\hbar\psi \end{cases}$$

Since each ψ is a common eigenfunction of $\hat{L}^2$, $\hat{H}$ and $\hat{L}_z$, within ψ there exists information about the corresponding eigenvalues. So we could better write $\psi_{E\ell m}$ for ψ, such that

$$\begin{cases} \hat{H}\psi_{E\ell m} = E\psi_{E\ell m} \\ \hat{L}^2\psi_{E\ell m} = \ell(\ell+1)\hbar^2\psi_{E\ell m} \\ \hat{L}_z\psi_{E\ell m} = m\hbar\psi_{E\ell m} \end{cases}$$

with $\ell = 0, 1, 2, \ldots$, $m = -\ell, -\ell+1, \ldots, \ell$

and $E \in \mathbb{R}$. The eigenfunctions $\psi_{E\ell m}$ are functions of r, θ, ϕ

$$\psi_{E\ell m} = \psi_{E\ell m}(r, \theta, \phi)$$

But we have already solved for the eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$ and have found

$$\hat{L}^2 Y_\ell^m(\theta,\phi) = \ell(\ell+1)\hbar^2 Y_\ell^m(\theta,\phi)$$

$$\hat{L}_z Y_\ell^m(\theta,\phi) = m\hbar Y_\ell^m(\theta,\phi)$$

And since $\hat{L}^2$ and $\hat{L}_z$ do not depend on r we can multiply $Y_\ell^m(\theta,\phi)$ by an arbitrary r function and still we have an eigenfunction of $\hat{L}^2$ and $\hat{L}_z$. Thus we put

$$\psi_{E\ell m}(r,\theta,\phi) = R(r)\, Y_\ell^m(\theta,\phi)$$

Now concerning the third equation $\hat{H}\psi_{E\ell m} = E\psi_{E\ell m}$ we have

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) - \frac{\hat{L}^2}{2mr^2} + V(r)\right] R Y_\ell^m = E R Y_\ell^m$$

$$-\frac{\hbar^2}{2m}\left(R'' + \frac{2}{r}R'\right) - \frac{\ell(\ell+1)\hbar^2}{2mr^2}R + (V(r) - E)R = 0$$

We see that each R may depend on E and ℓ (but not on m).

Noninteracting Particles and Separation of Variables

Consider a system described by two noninteracting particles 1 and 2. Let

$$q_1 = (x_1, y_1, z_1)$$
$$q_2 = (x_2, y_2, z_2)$$

represent the particles coordinates. Since the particles do not interact with each other, the classical Hamiltonian can be written as

$$H = \underbrace{T_1 + V_1}_{H_1} + \underbrace{T_2 + V_2}_{H_2}$$

leading, upon quantization, to the quantum Hamiltonian

$$\hat{H} = \hat{H}_1 + \hat{H}_2 .$$

The the time-ind. Schrödinger equation reads

$$\hat{H}\psi = E\psi$$

$$\left(\hat{H}_1 + \hat{H}_2\right)\psi(q_1, q_2) = E\,\psi(q_1, q_2)$$

Using the separation of variables technique we write

$$\psi(q_1, q_2) = G_1(q_1)\, G_2(q_2)$$

and obtain

$$(\hat{H}_1 + \hat{H}_2)\, G_1 G_2 = E\, G_1 G_2$$

$$G_2 \hat{H}_1 G_1 + G_1 \hat{H}_2 G_2 = E\, G_1 G_2$$

$$\underbrace{\frac{\hat{H}_1 G_1}{G_1}}_{E_1} + \underbrace{\frac{\hat{H}_2 G_2}{G_2}}_{E_2} = E$$

and separate into two ind. differential eqs

$$\begin{cases} \hat{H}_1 G_1 = E_1 G_1 \\ \hat{H}_2 G_2 = E_2 G_2 \end{cases}$$

Solving these two independent diff eqs amounts to finding the eigenfunctions G_1 and G_2 and the eigenvalues E_1 and E_2. Once we have done that, the solution to the original problem reads

$$\begin{cases} \psi(q_1, q_2) = G_1(q_1)\, G_2(q_2) \\ E = E_1 + E_2 \end{cases}$$

This result can be straightforwardly generalized to a system of m non interacting particles

$$\hat{H} = \sum_i \hat{H}_i$$

$$\psi(q_1, \dots, q_m) = \prod_i G_i(q_i)$$

$$E = \sum E_i$$

$$\hat{H}_i G_i = E_i G_i \quad , \quad i = 1, \dots, m$$

Also the same idea can be applied to a single particle whose Hamiltonian is the sum of independent terms for each coordinate

$$\left\{ \begin{array}{l} \hat{H} = \hat{H}_x(\hat{x}, \hat{p}_x) + \hat{H}_y(\hat{y}, \hat{p}_y) + \hat{H}_z(\hat{z}, \hat{p}_z) \\ \psi(x, y, z) = F(x) G(y) K(z) \\ E = E_x + E_y + E_z \\ \hat{H}_x F = E_x F \ , \ \hat{H}_y G = E_y G, \ \hat{H}_z K = E_z K \end{array} \right.$$