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Summary

Given an n-parameter Lie group of transformations on an m-dimensional space

$$x'_i = f_i(x_1, \dots, x_m; a_1, \dots, a_n)$$

the functions F on this space transform according to

$$dF = \frac{\partial F}{\partial x_j} dx_j$$

$$= \frac{\partial F}{\partial x_j}\left(\left.\frac{\partial f_j}{\partial a_i}\right|_{a_i=0} da_i \right)$$

$$= da_i \underbrace{\left(\left.\frac{\partial f_j}{\partial a_i}\right|_{a_i=0} \frac{\partial}{\partial x_j} \right)}_{X_i} F$$

By defining

$$X_i \equiv \left.\frac{\partial f_j}{\partial a_i}\right|_{a_i=0} \frac{\partial}{\partial x_j}$$

we calculate the structure constants

$$[X_i, X_j] = c_{ij}{}^k X_k$$

Irreducible Representations of SO(2) and SO(3)

Recall that in order to prove ~~the~~ Schur's Lemmas and the Orthogonality Theorem we needed the Rearrangement Theorem

$$\sum_g f(g) = \sum_g f(g'g)$$

For continuous groups, we shall need the analogous

$$\int f(R)\, dR = \int f(R'R)\, dR$$

In terms of the parameters a, we may rewrite

$$dR = g(R)\, da$$

where $g(R)$ represents the density of group elements in parameter space in the neighborhood of R.

Orthogonality of Characters for SO(2)

We claim that the rearrangement theorem for SO(2) reads

$$\int_0^{2\pi} R(\varphi')R(\varphi)\,d\varphi = \int_0^{2\pi} R(\varphi)\,d\varphi$$

proof:

$$\int_0^{2\pi} R(\varphi')R(\varphi)\,d\varphi = \int_0^{2\pi} R(\varphi'+\varphi)\,d\varphi \qquad \begin{array}{l}\theta = \varphi'+\varphi \\ d\theta = d\varphi\end{array}$$

$$= \int_{\varphi=0}^{\varphi=2\pi} R(\theta)\,d\theta$$

$$= \int_{\theta=\varphi'}^{\theta=\varphi'+2\pi} R(\theta)\,d\theta$$

$$= \int_{\varphi'}^{2\pi} R(\theta)\,d\theta + \int_{2\pi}^{2\pi+\varphi'} R(\theta)\,d\theta$$

$$= \int_{\varphi'}^{2\pi} R(\theta)\,d\theta + \int_0^{\varphi'} R(\theta)\,d\theta = \int_0^{2\pi} R(\theta)\,d\theta$$

Characters of Irreducible Representations

To begin with, we claim that all irreducible representations of an Abelian group are necessarily one dimensional.

In fact, let G be an Abelian group and consider

$$R : G \longrightarrow GL(V)$$

an irreducible representation of G on a finite complex vector space V. Take two matrices $R(a)$, $R(b)$ in the image of R.

Then we have

$$R(a)\,R(b) = R(ab) = R(ba) = R(b)\,R(a).$$

For a fixed $a \in G$, $R(a)$ commutes with all matrices of the irreducible representation R.

Therefore, by Schur's First Lemma, $R(a) = cI$. Since R is irreducible, R must be unidimensional.

Now, SO(2) is Abelian. Its irreducible representations are one dimensional. Each conjugacy class in SO(2) contains only one element. Thus, every element of SO(2) is in a class by itself and the characters are complex numbers which must satisfy the multiplication table of the group:

$$\chi(\varphi)\,\chi(\varphi') = \chi(\varphi + \varphi')$$

For the unity element we must have

$$\chi(0) = 1$$

By the properties of the group elements of SO(2), we must also have

$$\chi(\varphi + 2\pi) = \chi(\varphi)$$

Now consider the property

$$x(\varphi+\theta) = x(\varphi)\, x(\theta)$$

and calculate the derivative in both sides with respect to θ and evaluate at $\theta = 0$:

$$\frac{d}{d\theta}\left[x(\varphi+\theta)\right]\Big|_{\theta=0} = \frac{d}{d\theta}\left[x(\varphi)\, x(\theta)\right]\Big|_{\theta=0}$$

$$\left[\frac{d\,x(\varphi+\theta)}{d(\varphi+\theta)} \cdot \underbrace{\frac{d(\varphi+\theta)}{d\theta}}_{1}\right]\Big|_{\theta=0} = x(\varphi)\, \frac{d\,x(\theta)}{d\theta}\Big|_{\theta=0}$$

$$\frac{d\,x(\varphi+\theta)}{d(\varphi+\theta)}\Big|_{\theta=0} = x(\varphi)\, x'(0)$$

$$\frac{dx}{d\varphi} = x(\varphi)\, x'(0)$$

this is an ODE with solution

$$x(\varphi) = \underbrace{x(0)}_{1}\, e^{x'(0)\varphi}$$

Using the identity condition $x(0) = 0$, we get

$$\chi(\varphi) = e^{\chi'(0)\varphi}$$

The periodicity condition $\chi(\varphi+2\pi) = \chi(\varphi)$ now implies

$$e^{\chi'(0)\varphi} = e^{\chi'(0)(\varphi+2\pi)}$$

that means

$$e^{\chi'(0)2\pi} = 1$$

The most general solution amounts to

$$\chi'(0) = im \quad , \text{with} \quad m \in \mathbb{Z} .$$

Thus the character function of an irreducible representation reads

$$\chi(\varphi) = e^{im\varphi}$$

Since we have a different solution for each $m \in \mathbb{Z}$, the index m labels the different irreducible representations and we may write

$$\boxed{\chi_m(\varphi) = e^{im\varphi}}$$

orthogonality relations

Noticing that

$$\int_0^{2\pi} e^{i(m-m)\varphi}\, d\varphi = 2\pi\, \delta_{mm}$$

and

$$\chi_m(\varphi) = e^{im\varphi}$$

we write

$$\int_0^{2\pi} [\chi_m(\varphi)]^* [\chi_m(\varphi)]\, d\varphi = 2\pi\, \delta_{mm}$$

$$\int_0^{2\pi} d\varphi = 2\pi$$

We call $2\pi = \int_0^{2\pi} d\varphi$ the "volume" of the group in the space of the parameter φ. (Recall the orthogonality theorem for characters in the case of finite groups: $\sum_\alpha m_\alpha \chi_\alpha^k \chi_\alpha^{k'*} = |G|\,\delta_{k,k'}$)

Example:

Consider the following representation of $SO(2)$:

$$R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

Since $SO(2)$ is Abelian, this representation must be reducible.

Since we have

$$\chi(\varphi) = \mathrm{tr}\left[R(\varphi)\right]$$
$$= 2\cos\varphi = e^{i\varphi} + e^{-i\varphi} = \chi_1(\varphi) + \chi_{(-1)}(\varphi)$$

this representation is the direct sum of the irreducible representations $\chi_1(\varphi)$ and $\chi_{(-1)}(\varphi)$

In fact, by chosing for instance

$$A = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

we have $A^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ -i/2 & i/2 \end{pmatrix}$ and

$$A\,R(\varphi)\,A^{-1} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} 1/2 & 1/2 \\ -i/2 & i/2 \end{pmatrix} = \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$$

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Basis Functions for Irreducible Representations

Considering the SO(2) representation

$$R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

we calculate the eigenvalues

$$\det(R - \lambda I) = \begin{vmatrix} \cos\varphi - \lambda & -\sin\varphi \\ \sin\varphi & \cos\varphi - \lambda \end{vmatrix}$$

$$= (\cos\varphi - \lambda)^2 + \sin^2\varphi$$

$$= 1 - 2\lambda\cos\varphi + \lambda^2$$

$$\boxed{\lambda^2 - 2\lambda\cos\varphi + 1 = 0}$$

$$\Delta = 4\cos^2\varphi - 4 = 4(\cos^2\varphi - 1) = -4\sin^2\varphi$$

$$\lambda = \frac{2\cos\varphi \pm 2i\sin\varphi}{2} = \cos\varphi \pm i\sin\varphi = e^{\pm i\varphi}$$

$$\boxed{\lambda_1 = e^{i\varphi}, \quad \lambda_2 = e^{-i\varphi}}$$

eigenvectors

$$R(\varphi)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = e^{i\varphi}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} \cos\varphi\, x - \sin\varphi\, y = e^{i\varphi} x \\ \sin\varphi\, x + \cos\varphi\, y = e^{i\varphi} y \end{cases}$$

$$\begin{cases} \cancel{\cos\varphi\, x} - \sin\varphi\, y = \cancel{\cos\varphi\, x} + i\sin\varphi\, x \\ \sin\varphi\, x + \cancel{\cos\varphi\, y} = \cancel{\cos\varphi\, y} + i\sin\varphi\, y \end{cases}$$

For $\varphi \neq n\pi$:

$$\begin{cases} -y = ix \\ x = iy \end{cases}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -ix \end{pmatrix} = x\begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad \boxed{v_1 = x\begin{pmatrix} 1 \\ -i \end{pmatrix}}$$

$$R(\varphi)\begin{pmatrix} x \\ y \end{pmatrix} = e^{-i\varphi}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{matrix} \cancel{\cos\varphi\, x} - \sin\varphi\, y = \cancel{\cos\varphi\, x} - i\sin\varphi\, x \\ \sin\varphi\, x + \cancel{\cos\varphi\, y} = \cancel{\cos\varphi\, y} - i\sin\varphi\, y \end{matrix}$$

$(\varphi \neq n\pi)$

$$\Rightarrow \begin{cases} y = ix \\ x = -iy \end{cases} \qquad \boxed{v_2 = \begin{pmatrix} x \\ ix \end{pmatrix} = x\begin{pmatrix} 1 \\ i \end{pmatrix}}$$

In the space of $\sqrt{\text{complex analytic}}$ functions:

$$F : \mathbb{R}^2 \longrightarrow \mathbb{C} \quad ,$$

the transf. $R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

induces the transf.

$$dF = \frac{\partial F}{\partial x_j} dx_j$$

$$= \frac{\partial F}{\partial x_j} \left(\left.\frac{\partial f_j}{\partial \varphi}\right|_{\varphi=0} d\varphi \right)$$

$$= \underbrace{d\varphi \left(\left.\frac{\partial f_j}{\partial \varphi}\right|_{\varphi=0} \right) \frac{\partial}{\partial x_j}}_{X} F$$

$$X = \left(\frac{\partial f_j}{\partial \varphi} \right)\Bigg|_{\varphi=0} \frac{\partial}{\partial x_j} \longrightarrow \text{generator of } SO(2)$$

$$\begin{cases} x' = \cos\varphi\, x - \sin\varphi\, y & \to f_1(x,y,\varphi) = \cos\varphi\, x - \sin\varphi\, y \\ y' = \sin\varphi\, x + \cos\varphi\, y & \to f_2(x,y,\varphi) = \sin\varphi\, x + \cos\varphi\, y \end{cases}$$

$$\left(\frac{\partial f_1}{\partial \varphi}\right)\Big|_{\varphi=0} = \left(-\sin\varphi\, x - \cos\varphi\, y\right)\Big|_{\varphi=0} = -y$$

$$\left(\frac{\partial f_2}{\partial \varphi}\right)\Big|_{\varphi=0} = \left(\cos\varphi\, x - \sin\varphi\, y\right)\Big|_{\varphi=0} = x$$

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

vector space considered: space of complex functions $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$

For the eigenvalue $\lambda_1 = e^{i\varphi}$, we look for eigenfunctions $f(x,y)$ satisfying

$$e^{\varphi X} f(x,y) = e^{i\varphi} f(x,y)$$

that is

$$e^{\varphi\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)} f(x,y) = e^{i\varphi} f(x,y)$$

By the change of variables

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

we have

$$\begin{cases} \dfrac{\partial}{\partial\theta} = \dfrac{\partial x}{\partial\theta}\dfrac{\partial}{\partial x} + \dfrac{\partial y}{\partial\theta}\dfrac{\partial}{\partial y} \\[2ex] \dfrac{\partial}{\partial r} = \dfrac{\partial x}{\partial r}\dfrac{\partial}{\partial x} + \dfrac{\partial y}{\partial r}\dfrac{\partial}{\partial y} \end{cases}$$

which leads to

$$\frac{\partial}{\partial\theta} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Thus, considering $\tilde{f}(r,\theta) \equiv f(x(r,\theta), y(r,\theta))$,

we must solve

$$e^{\varphi\frac{\partial}{\partial\theta}}\tilde{f}(r,\theta) = e^{i\varphi}\tilde{f}(r,\theta)$$

Expanding the exponential:

$$\tilde{f}(r,\theta) = \tilde{f}(r,\theta) \qquad \varphi^2\frac{\partial^2}{\partial\theta^2}\tilde{f}(r,\theta) = -\varphi^2\tilde{f}(r,\theta)$$

$$\varphi\frac{\partial}{\partial\theta}\tilde{f}(r,\theta) = i\varphi\,\tilde{f}(r,\theta) \qquad \varphi^3\frac{\partial^3}{\partial\theta^3}\tilde{f}(r,\theta) = -i\varphi^3\tilde{f}(r,\theta)$$

these eqs are all consequences from

$$\frac{\partial}{\partial \theta} \tilde{f}(r,\theta) = i \tilde{f}(r,\theta)$$

whose solution reads

$$\tilde{f}(r,\theta) = e^{i\theta} g(r)$$

Back to $f(x,y)$ we get

$$f(x,y) = (x + iy)\, h(x^2+y^2)$$

Following the ideas of Vvdensky ...

The eigenvectors of $R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

corresponding to eigenvalues $e^{\pm i\varphi}$ are proportional

to $x \pm iy$. Operating on these vectors with $R(\varphi)$,

we get

$$\begin{cases} x' = x\cos\varphi - y\sin\varphi \\ y' = x\sin\varphi + y\cos\varphi \end{cases}$$

$$x' \pm iy' = x(\cos\varphi \pm i\sin\varphi) \pm iy(\cos\varphi \pm i\sin\varphi)$$

$$= (x \pm iy)(e^{\pm i\varphi})$$

Considering functions of the form $(x \pm iy)^m$

with $m \in \mathbb{Z}$, we have

$$(x' \pm iy')^m = (x \pm iy)^m e^{\pm im\varphi}$$

We call $(x \pm iy)^m$ the basis functions,

and the character table reads

	E	$R(\varphi)$	
$\Gamma^{\pm m}$:	1	$e^{\pm im\varphi}$	$(x \pm iy)^m$

Reviewing the Regular Representation

- Case of Finite Groups

The regular rep. maybe defined as the rep. in which the <u>group acts on itself</u>. Given a finite group, we may construct a vector space V over $\mathbb{C}$ consisting of all linear combinations of the group elements. The group elements form themselves a basis. Now this vector space is made naturally into an algebra by the group multiplication law.

<u>Example</u>: $G = D_3 = \{e, a, b, a^2, ab, a^2b\}$

$$V = \{\alpha_1 e + \alpha_2 a + \alpha_3 b + \alpha_4 a^2, \alpha_5 ab, \alpha_6 a^2 b;\ \alpha_i \in \mathbb{C}\}$$

A typical $v \in V$ may be written

$$v = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_6 \end{pmatrix}, \quad \text{with } \alpha_i \in \mathbb{C}$$

These are the v components in the natural basis $B = \{e, a, b, a^2, ab, a^2b\}$

Now the reg. rep. is defined as

$$\begin{aligned} \Pi : D_3 &\longrightarrow GL(V) \\ g &\longmapsto g : V \to V \\ & \qquad\quad v \mapsto gv \end{aligned}$$

Choosing the natural basis $B = \{e, a, b, a^2, ab, a^2b\}$ the operators in ~~the~~ $\mathrm{Im}(\Pi)$ are rep. by 6×6 matrices:

$$e = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$$\begin{aligned} a : \quad ae &= 0e + 1a + 0b + 0a^2 + 0ab + 0a^2b \\ aa &= 0 + 0 + 0 + 1a^2 + 0 + 0 \\ ab &= 0 + 0 + 0 + 0 + 1ab + 0 \\ aa^2 &= 1e + 0 + 0 + 0 + 0 + 0 \\ aab &= 0 + 0 + 0 + 0 + 0 + 1a^2b \\ aa^2b &= 0 + 0 + 1b + 0 + 0 + 0 \end{aligned}$$

$$a \to \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & & & & \\ 0 & 0 & & & & \\ 0 & 0 & & & & \end{pmatrix}$$

We know the regular representation contains all the irreducible representations of the group.

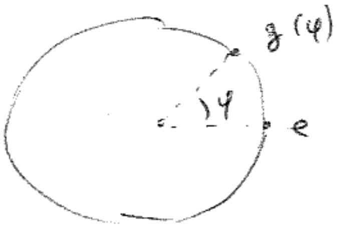
Now, the previous vector space V of all linear combinations of the group elements may be thought as the space of all complex functions defined on the group G. Any $v \in V$ defines a complex function on G, which associates to any $g \in G$ the amount of g in the linear combination (this can be also achieved by the canonical inner product in V)

Back to the present case of Lie groups, consider now $G = SO(2)$ and the space $L^2(S^1)$ of complex functions

$$f : SO(2) \longrightarrow \mathbb{C}$$

which can be expanded in Fourier series

$$f(\varphi) = \sum_{m=-\infty}^{m=+\infty} a_m e^{im\varphi}$$



This is a vector space of infinite dimension.

A basis for such space is $B = \{ e^{im\varphi} ; m \in \mathbb{Z} \}$

Similarly to the finite case, we consider the action of the group $SO(2)$ on itself, that is, we consider the "regular representation":

$$T : SO(2) \longrightarrow GL(L^2(S^1))$$
$$\varphi \longmapsto T_\varphi$$

with $T_\varphi : L^2(S^1) \longrightarrow L^2(S^1)$
$$f(\theta) \longmapsto T_\varphi f(\theta)$$

We know the regular rep. is reducible, and from our previous calculations, we may write

$$T_\varphi = \begin{pmatrix} 1 & & & & & \\ & e^{i\varphi} & & & & \\ & & e^{-i\varphi} & & & \\ & & & e^{2i\varphi} & & \\ & & & & e^{-2i\varphi} & \\ & & & & & \ddots \end{pmatrix}$$

for the matrix for the reg. rep. in the basis $\mathcal{B} = \{ 1, e^{i\theta}, e^{-i\theta}, e^{2i\theta}, \ldots \}$

Expanding $f(\theta)$ in this basis

$$f(\theta) = \sum_{m=-\infty}^{m=+\infty} a_m e^{im\theta}$$

the action of T_φ is written as

$$T_\varphi f(\theta) = \begin{pmatrix} 1 & & & & \\ & e^{i\varphi} & & & \\ & & e^{-i\varphi} & & \\ & & & e^{2i\varphi} & \\ & & & & \ddots \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_{-1} \\ a_2 \\ \vdots \end{pmatrix}$$

and naturally we have

$$T_\varphi f(\theta) = f(\theta + \varphi)$$

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Axis-Angle Representation of Proper Rotations in Three Dimensions

Eigenvalues of Orthogonal Matrices

Consider A a proper rotation matrix in three dimensions. That means $A \in M_3(\mathbb{R})$ with $A^T A = I$ and $\det A = 1$. Let u_i and λ_i denote its eigenvectors and eigenvalues

$$A u_i = \lambda_i u_i$$

Although the entries of A are real, we consider this eigenvalue equation in $\mathbb{C}^3$ over $\mathbb{C}$, which means $u_i \in \mathbb{C}^3$, $\lambda_i \in \mathbb{C}$ and A is a matrix representing a linear operator in $L(\mathbb{C}^3)$.

Then we have

$$A u_i = \lambda_i u_i \longrightarrow u_i^\dagger A^\dagger = \lambda_i^* u_i^\dagger$$

$$\rightarrow u_i^\dagger u_i = u_i^\dagger A^T A u_i = u_i^\dagger A^\dagger A u_i = |\lambda_i|^2 u_i^\dagger u_i$$

Note that, since the entries of A are real, we have $A^\dagger = A^T$.

Since $u_i \neq 0$, we have $|\lambda_i|^2 = 1$.

The characteristic polynomial $\det(A - \lambda I) = 0$ is a third-degree polynomial with real coefficients. The solutions are

$$\lambda_1 = 1, \quad \lambda_2 = e^{i\varphi}, \quad \lambda_3 = e^{-i\varphi}$$

The eigenvector corresponding to $\lambda_1 = 1$ defines the axis of rotation and φ defines the rotation angle.

Consider the axis of rotation to be defined by a unit vector $\hat{n}$. Then we have

$$A\hat{n} = \hat{n}$$

from which we get

$$A^T\hat{n} = A^T A\hat{n} = \hat{n}$$

and thus

$$(A - A^T)\hat{n} = 0$$

The equation $(A - A^T)\hat{m} = 0$ plus the normalization condition $\|\hat{m}\|^2 = 1$ determines the real vector $\hat{m}$ uniquely

(The rotation angle can be determined by the trace invariance: $\operatorname{tr} A = 1 + e^{i\varphi} + e^{-i\varphi} = 1 + 2\cos\varphi$)

Normal Form of an Orthogonal Matrix

Given a real matrix 3×3 matrix $A \in SO(3)$ we know A represents a proper rotation (and vice-versa). Considering the field of complex numbers $\mathbb{C}$ and the vector space $\mathbb{C}^3$ over $\mathbb{C}$, the matrix A can be diagonalized to

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\varphi} & 0 \\ 0 & 0 & e^{-i\varphi} \end{pmatrix}$$

Now $\Lambda \in SU(3)$. The eigenvectors can be chosen orthonormal, constituting a basis for $\mathbb{C}^3$ as

$$\hat{m} = \hat{m}_1 = (1, 0, 0)$$

$$\hat{m}_2 = (0, 1, 0)$$

$$\hat{m}_3 = (0, 0, 1)$$

In order to come back to $SO(3)$ we perform a change of basis.

By chosing

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & i\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} & -i\frac{\sqrt{2}}{2} \end{pmatrix}$$

we get

$$R = U^{-1} \Lambda U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

The "price" amounts to changing the basis to

$$\hat{m} = m_1 = (1, 0, 0)$$

$$\hat{m}_2 = \frac{\sqrt{2}}{2}(0, 1, i)$$

$$\hat{m}_3 = \frac{\sqrt{2}}{2}(0, 1, -i)$$

Parameter Space for SO(3)

A convenient parametrization for SO(3) which emerges from the axis-angle realization of rotations is the following:

$$(\hat{m}, \varphi) \longrightarrow (m_1\varphi, m_2\varphi, m_3\varphi)$$

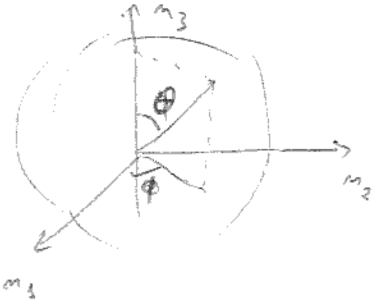
with $m_1^2 + m_2^2 + m_3^2 = 1$ and $0 \leq \varphi \leq \pi$.

For each fixed value of φ, we have

$$(m_1\varphi)^2 + (m_2\varphi)^2 + (m_3\varphi)^2 = \varphi^2$$

and we have a sphere of radius φ.

We may define spherical coordinates φ, ϕ, θ, such that



$$\begin{cases} m_1\varphi = \varphi\cos\phi\sin\theta \\ m_2\varphi = \varphi\sin\phi\sin\theta \\ m_3\varphi = \varphi\cos\theta \end{cases}$$

Now SO(3) can be viewed geometrically as the interior of a sphere of radius $\varphi = \pi$. Recall that SO(3) is three-dimensional.

Notice that two diametrically opposed points on the surface of the sphere correspond to the same element in $SO(3)$.

Two elements in $SO(3)$ are in the same equivalence class if they correspond to rotations with the same angle φ. This can be seen from

$$R(\hat{m},\varphi) = \left[U(\hat{m},\hat{m}') \right]^{-1} R(\hat{m}',\varphi) \left[U(\hat{m},\hat{m}') \right]$$