C 21 03/05/23 Thibes

Continuous Groups, Lie Groups and Lie Algebras

- Continuous Groups

Consider an element R which depends on n real continuous parameters

$$R(a) \equiv R(a_1, a_2, \dots, a_n)$$

We have a continuous group if the R's fulfill the group requirements and if there is some notion of proximity or continuity in the sense that a small change in one of the factors induces a correspondingly small change in their product.

$$R(c) = R(a)\, R(b)$$

$$c = f(a, b)$$

From the associativity of the composition law

$$R(a) \Big[\underbrace{R(b)\, R(c)}_{R(f(b,c))} \Big] = \Big[\underbrace{R(a)\, R(b)}_{R(f(a,b))} \Big] R(c)$$

$$R(a)\left[\underbrace{R(b)R(c)}_{R(f(b,c))}\right] = \left[\underbrace{R(a)R(b)}_{R(f(a,b))}\right] R(c)$$

we get

$$R\big(f(a, f(b,c))\big) = R\big(f(f(a,b), c)\big)$$

From injectivity

$$f\big(a, f(b,c)\big) = f\big(f(a,b), c\big)$$

Existence on ~~an~~ identity element, $R(a_0)$, ~~such~~ ~~to~~ such that

$$R(a_0)R(a) = R(a)R(a_0) = R(a)$$

leads to

$$f(a_0, a) = f(a, a_0) = a$$

Similarly, if $R(a')$ denotes the inverse of $R(a)$, we have

$$R(a')R(a) = R(a)R(a') = R(a_0)$$

which means

$$f(a', a) = f(a, a') = a_0$$

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If f is an analytic function, that is a function with a convergent Taylor series ~~within~~ within the domain defined by the parameters, the group is called an

n-parameter Lie group

Group Transformations on d-dimensional spaces

We may consider Minkowski spaces, Euclidean spaces, where the variables are space-time coordinates or spaces associated with internal degrees of freedom such as spin or isospin.

Example:

$$T_a : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto x' = ax$$

for a fixed $a \in \mathbb{R}$ with $a \neq 0$.

$$T_a : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto x' = ax$$

$$(T_a \circ T_b)x = T_a(T_b x)$$
$$= T_a(bx)$$
$$= abx$$
$$= (T_{ab})x$$

In terms of the function f, we have

$$T_a \circ T_b = T_{ab}$$

which means

$$c = f(a,b) = ab$$

the inverse is given by

$$a' = a^{-1}$$

and $a = 1$ corresponds to the neutrum element.

Hence we have a one parameter Abelian Lie group.

Example 2:

$$T_{(a_1,a_2)} : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto x' = a_1 x + a_2$$

Notice that

$$T_{(a_1,a_2)} \circ T_{(b_1,b_2)} (x) =$$

$$= T_{(a_1,a_2)} \left(b_1 x + b_2 \right)$$

$$= a_1 (b_1 x + b_2) + a_2$$

$$= a_1 b_1 x + (a_1 b_2 + a_2)$$

and thus, we have

$$c = f(a, b)$$

$$\begin{cases} c_1 = a_1 b_1 \\ c_2 = a_1 b_2 + a_2 \end{cases}$$

In this case we have a two-parameter non-Abelian Lie group.

In example 2, we have an isomorphism to ~~the~~ a group of 2×2 matrices:

$$\begin{pmatrix} a_1 & a_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 x + a_2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a_1 & a_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b_1 & b_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_1 b_1 & a_1 b_2 + a_2 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a_1 & a_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_1^{-1} & -a_2 a_1^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Lie Groups

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In order to have a precise definition for a Lie Group, we first define the concepts of topological space and differentiable manifold.

Topological Space

Let X be a set and $\tau = \{U_i ; i \in I\}$ a collection of subsets of X. The pair (X, τ) is said to be a topological space when

(i) $\emptyset, X \in \tau$

(ii) If J is any subset of I then

$$\bigcup_{j \in J} U_j \in \tau$$

(iii) If K is a finite subset of I then

$$\bigcap_{h \in K} U_h \in \tau$$

Differentiable Manifold

We say M is an m-dimensional manifold when

(i) M is a topological space

(ii) M is provided with a family of pairs $\{(U_i, \varphi_i)\}$

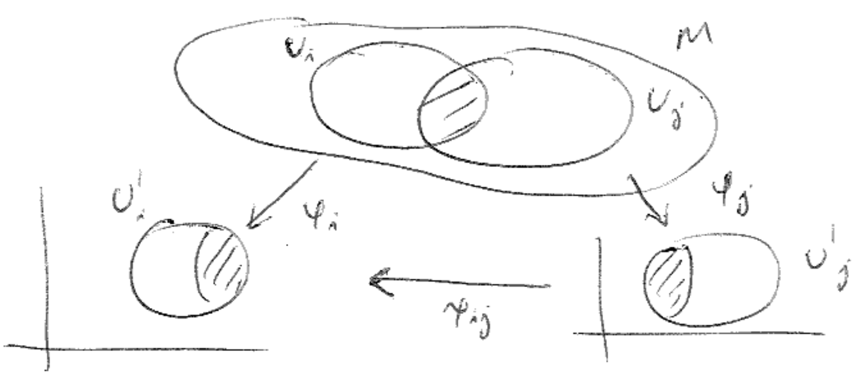
(iii) $\{U_i\}$ is a family of open sets such that

$$\bigcup_i U_i = M$$

and φ_i is a homeomorphism from U_i to an open subset U_i' of $\mathbb{R}^m$

$$\varphi_i : U_i \longrightarrow U_i'$$

(iv) Given U_i, U_j with $U_i \cap U_j \neq \emptyset$, the map $\psi_{ij} = \varphi_i \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ is infinitely differentiable



Lie Group

Def.: A Lie Group is a differentiable manifold endowed with a group structure such that the group operations

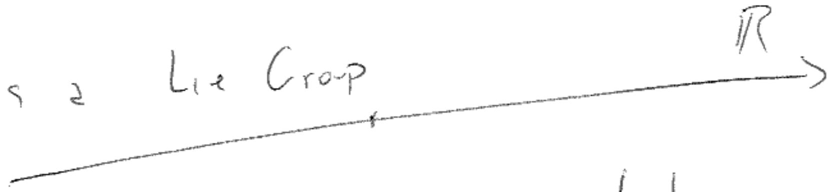
$$\text{(i)} \quad \cdot : G \times G \longrightarrow G$$
$$(g_1, g_2) \longmapsto g_1 g_2$$

$$\text{(ii)} \quad {}^{-1} : G \longrightarrow G$$
$$g \longmapsto g^{-1}$$

are differentiable

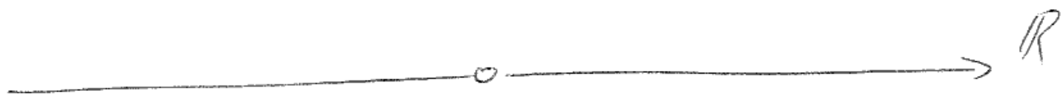
Examples:

Ex 0: $(\mathbb{R}, +)$ is a Lie Grop



connected

Ex 0': $(\mathbb{R}^{\times}, \cdot)$ is a Lie Group



not connected

Ex 1: $O(2) = \{ A \in GL(2,\mathbb{R}),\ A^T I A = I \}$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Take
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and impose the conditions:

$$A^T I A = I \longrightarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A I A^T = I \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{cases} \boxed{a^2+b^2=1} \\ ac+bd=0 \\ c^2+d^2=1 \end{cases} \qquad \begin{cases} a^2+c^2=1 & \longrightarrow \quad c = \pm b \\ ab+cd=0 \\ b^2+d^2=1 & \longrightarrow \quad d = \pm a \end{cases}$$

If $c = b$, then $ab + cd = (a+d)b = 0$.

If $b \neq 0$, $\boxed{d = -a}$

~~If $b = 0$, $a \neq 0$ and $ac+bd = a+d$~~

If $c = -b \neq 0$, then $ab - bd = 0$ and $d = a$

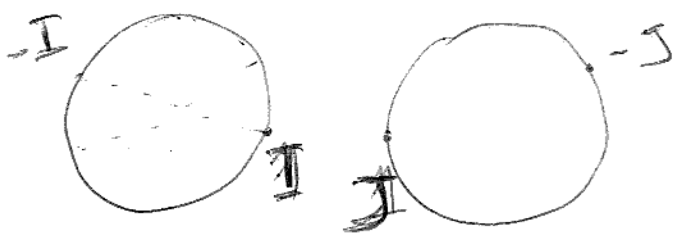
If $b = 0$, $a = \pm 1$ and $d = \pm 1$

therefore we have the two possibilities

$$A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

with $a^2 + b^2 = 1$ $\qquad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$O(2)$:



this is a <u>noncommutative nonconnected compact Lie group.</u>

Note that $\{I, -I\}$, $\{I, J\}$, $\{I, -J\}$ and $\{I, J, -I, -J\}$ are subgroups of $O(2)$. However, from these four, only $\{I, -I\}$ is a normal subgroup.

Ex. 2:

$$O(1,1) = \{ A \in GL(2,\mathbb{R}) ;\ AJA^T = J \}$$

with $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{cases} \boxed{a^2 - b^2 = 1} \\ ac - bd = 0 \\ c^2 - d^2 = -1 \end{cases} \qquad \begin{cases} a^2 - c^2 = 1 & \to c = \pm b \\ ab - cd = 0 \\ b^2 - d^2 = -1 & \to d = \pm a \end{cases}$$

If $c = b \neq 0$:

$$ab - bd = 0 \to \boxed{d = a}$$

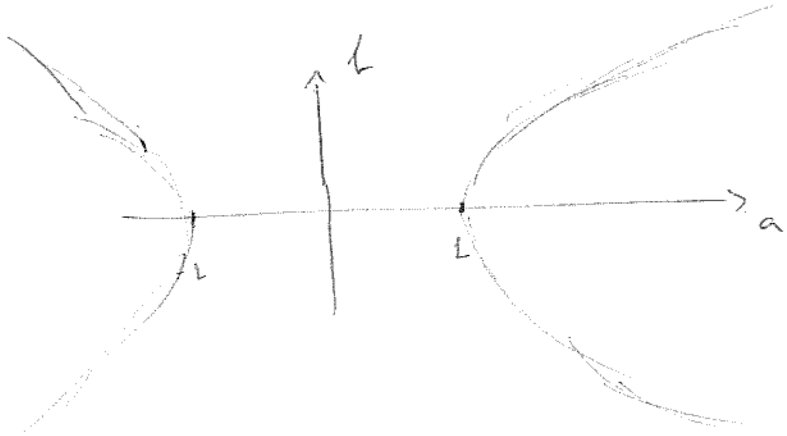
If $c = -b \neq 0$:

$$-ab - bd = 0 \to \boxed{d = -a}$$

If $b = 0$, then $c = 0$, $a = \pm 1$ and $d = \pm 1$.

Therefore, if $A \in O(1,1)$ it follows

$$A = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} a & b \\ -b & -a \end{pmatrix}$$



Linear Transformation Groups

We consider the group of linear transformations in d dimensions

$$x' = Ax \quad , \quad x, x' \in \mathbb{R}^d , \quad A \in GL(\mathbb{R}, d)$$

Example: $d = 2$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

with $\det A = a_{11}a_{22} - a_{12}a_{21} \neq 0$

In general, $GL(n, \mathbb{R})$ is a Lie group with n^2 parameters. If F is a field, we may generalize further to

$$GL(n, F)$$

For instance, for $GL(n, \mathbb{C})$ we have a Lie group of $2n^2$ parameters. However, it is not for every field which we have a Lie group. For instance, $GL(n, \mathbb{Q})$ is not a Lie group.

Orthogonal Groups

In ordinary Euclidian space, by imposing the preservation of lengths

$$\sum x_i'^2 = \sum x_i^2$$

we get the orthogonal group

$$O(n)$$

The matrices of this group are defined to satisfy

$$A^T A = I$$

Since $A^T A$ is symmetric, this amounts to imposing only $\frac{n^2}{2} + \frac{n}{2} = \frac{n(n+1)}{2}$ relations. Therefore the dimension of $O(n)$ is

$$= 160 = \qquad n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2} \quad //$$

Please notice that the matrices from $O(n)$ have real entries, by definition.

If we impose further $\det A = 1$, we get the Lie group $SO(n)$. But since we have

$$A^T A = I \rightarrow (\det A)^2 = 1 \rightarrow \det A = \pm 1$$

this condition does not change the dimension. Therefore

$$\dim SO(n) = \dim O(n) = \frac{n(n-1)}{2}$$

$O(1) \rightarrow 0$

$O(2) \rightarrow 1$

$O(3) \rightarrow 3$

$O(4) \rightarrow 6$

$O(5) \rightarrow 10$

$O(6) \rightarrow 15$

$\vdots$

For $A \in SO(2)$ we may write

$$A = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

with $\varphi \in \mathbb{R}$.

Infinitesimal Generators

Let us consider the SO(2) example.

The Taylor expansion gives us

$$R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

$$R(\varphi) = R(0) + \left.\frac{dR}{d\varphi}\right|_{\varphi=0} \varphi + \frac{1}{2} \left.\frac{d^2R}{d\varphi^2}\right|_{\varphi=0} \varphi^2 + \dots$$

$$\left.\frac{dR}{d\varphi}\right|_{\varphi=0} = \left.\begin{pmatrix} -\sin\varphi & -\cos\varphi \\ \cos\varphi & -\sin\varphi \end{pmatrix}\right|_{\varphi=0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\left.\frac{d^2R}{d\varphi^2}\right|_{\varphi=0} =$$

$$\left.\frac{d^3R}{d\varphi^3}\right|_{\varphi=0} =$$

$$\left.\frac{d^4R}{d\varphi^4}\right|_{\varphi=0} =$$

$$R(\varphi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\varphi + \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 \varphi^2 + \frac{1}{3!}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^3 \varphi^3 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{X^n \varphi^n}{n!} = \sum \frac{(X\varphi)^n}{n!}$$

$$\boxed{R(\varphi) = e^{\varphi X}}$$

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \longrightarrow \text{generator}$$

Notice that

$$X = \frac{dR}{d\varphi}(\varphi)\Big|_{\varphi=0}$$

and also

$$e^{\varphi X} = I\cos\varphi + X\sin\varphi$$

Alternatively, from the property

$$R(\varphi_1 + \varphi_2) = R(\varphi_1) R(\varphi_2)$$

we may write

$$R(\varphi + \varphi_2) = R(\varphi) R(\varphi_2)$$

$$\frac{d}{d\varphi_2} \left\{ R(\varphi + \varphi_2) \right\} \Big|_{\varphi_2 = 0} = \frac{d}{d\varphi_2} \left\{ R(\varphi) R(\varphi_2) \right\}_{\varphi_2 = 0}$$

$$\left\{ \frac{d R(\varphi+\varphi_2)}{d(\varphi+\varphi_2)} \cdot 1 \right\}_{\varphi_2 = 0} = R(\varphi) \frac{d R(\varphi_2)}{d\varphi_2} \Big|_{\varphi_2 = 0}$$

$$\frac{d R(\varphi)}{d\varphi} = \underbrace{\frac{dR}{d\varphi}\Big|_{\varphi=0}}_{X} R(\varphi) \qquad X \equiv \frac{dR}{d\varphi}\Big|_{\varphi=0}$$

$$\boxed{\frac{dR(\varphi)}{d\varphi} = X R(\varphi)}$$

solution: $\boxed{R(\varphi) = R(0)\, e^{X\varphi}}$

Since $R(0) = I$, we have $\boxed{R(\varphi) = e^{X\varphi}}$ as before

Operator Form of Generators

Consider the transformation

$$\begin{cases} x' = x\cos\varphi - y\sin\varphi \\ y' = x\sin\varphi + y\cos\varphi \end{cases}$$

For an infinitesimal angle $\varphi \longrightarrow d\varphi$ we have

$$\begin{cases} x' = x - y\,d\varphi \\ y' = x\,d\varphi + y \end{cases}$$

then

$$F(x', y') = F(x - y\,d\varphi,\ x\,d\varphi + y)$$

On the other side, we have

$$F(x', y') = F(x, y) + \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy$$

By using $dx = -y\,d\varphi$ and $dy = x\,d\varphi$, we write

$$F(x', y') = F(x, y) + d\varphi\left(-y\frac{\partial F}{\partial x} + x\frac{\partial F}{\partial y}\right)$$

$$F(x',y') = F(x,y) + d\varphi \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) F$$

We see the differential operator

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

generates the group.

SO(3)

SO(3) is the group of proper rotations in 3 dimensions. That means real orthogonal matrices of determinant one. We know this is a Lie group of dimension $\frac{3.(3-1)}{2} = 3$. Therefore we need three continuous real parameters.

Common choices are

- 3 successive rotations about fixed orthogonal axes
- Euler angles
- Cayley-Klein angles
- axis-angle rep.
- quaternions

Rotation Matrices

$$R_3(\varphi_3) = \begin{pmatrix} \cos\varphi_3 & -\sin\varphi_3 & 0 \\ \sin\varphi_3 & \cos\varphi_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_1(\varphi_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_1 & -\sin\varphi_1 \\ 0 & \sin\varphi_1 & \cos\varphi_1 \end{pmatrix}$$

$$R_2(\varphi_2) = \begin{pmatrix} \cos\varphi_2 & 0 & \sin\varphi_2 \\ 0 & 1 & 0 \\ -\sin\varphi_2 & 0 & \cos\varphi_2 \end{pmatrix}$$

generators:

$$X_1 = \left.\frac{dR_1}{d\varphi_1}\right|_{\varphi_1=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$X_2 = \left.\frac{dR_2}{d\varphi_2}\right|_{\varphi_2=0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$X_3 = \left.\frac{dR_3}{d\varphi_3}\right|_{\varphi_3=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Notice that the matrices

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad X_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

form a basis for the vector space of antisymmetric 3×3 matrices over $\mathbb{R}$. This vector space is also an algebra.

Note that

$$[X_i, X_j] = \epsilon_{ijk} X_k$$

Similarly, the differential operators can be written as

$$X_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$$

$$X_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$$

$$X_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

These can be obtained by considering three successive small rotations in the directions X, Y, Z. Recall that infinitesimal rotations commute. Therefore, to first order in the angles we have

$$R_1(\varphi_1)R_2(\varphi_2)R_3(\varphi_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\varphi_1 \\ 0 & \varphi_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & \varphi_2 \\ 0 & 1 & 0 \\ -\varphi_2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\varphi_3 & 0 \\ \varphi_3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \varphi_2 \\ 0 & 1 & -\varphi_1 \\ -\varphi_2 & \varphi_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -\varphi_3 & 0 \\ \varphi_3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\varphi_3 & \varphi_2 \\ \varphi_3 & 1 & -\varphi_1 \\ -\varphi_2 & \varphi_1 & 1 \end{pmatrix}$$

which permits us to write ~~a~~ general infinitesimal rotation as

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & -\varphi_3 & \varphi_2 \\ \varphi_3 & 1 & -\varphi_1 \\ -\varphi_2 & \varphi_1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

or

$$x' = x \underbrace{-\varphi_3 y + \varphi_2 z}_{dx}$$

$$y' = y \underbrace{+\varphi_3 x - \varphi_1 z}_{dy}$$

$$z' = z \underbrace{-\varphi_2 x + \varphi_1 y}_{dz}$$

$$\begin{cases} dx = -\varphi_3 y + \varphi_2 z \\ dy = \varphi_3 x - \varphi_1 z \\ dz = -\varphi_2 x + \varphi_1 y \end{cases}$$

By substituting in the differential for a function $F(x, y, z)$ we get

$$F = F(x, y, z)$$

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz$$

$$dF = \frac{\partial F}{\partial x} (-\varphi_3 y + \varphi_2 z) + \frac{\partial F}{\partial y} (\varphi_3 x - \varphi_1 z) +$$

$$+ \frac{\partial F}{\partial z} (-\varphi_2 x + \varphi_1 y)$$

$$dF = \varphi_1 \left(y \frac{\partial F}{\partial z} - z \frac{\partial F}{\partial y} \right) + \varphi_2 \left(z \frac{\partial F}{\partial x} - x \frac{\partial F}{\partial z} \right)$$

$$+ \varphi_3 \left(x \frac{\partial F}{\partial y} - y \frac{\partial F}{\partial x} \right)$$

recall $\varphi_1, \varphi_2, \varphi_3$ here were assumed to be infinitesimal. Writing more properly $d\varphi_1, d\varphi_2, d\varphi_3$ we get

$$dF = \left[d\varphi_1 \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) + d\varphi_2 \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right) + \right.$$

$$\left. + d\varphi_3 \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \right] F$$

$$= \left[d\varphi_1 X_1 + d\varphi_2 X_2 + d\varphi_3 X_3 \right] F$$

We may now compare with the QM angular momentum operators

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times (-i\hbar \vec{\nabla})$$

$$L_1 = -i\hbar \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) = -i\hbar X_1$$

$$L_2 = -i\hbar \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right) = -i\hbar X_2$$

$$L_3 = -i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) = -i\hbar X_3$$

From

$$[X_i, X_j] = \epsilon_{ijk} X_k$$

we get

$$[L_i, L_j] = (-i\hbar)^2 [X_i, X_j]$$

$$= -\hbar^2 \epsilon_{ijk} X_k$$

$$= -\hbar^2 \epsilon_{ijk} \left(\frac{i}{\hbar} \right) L_k$$

$$= -i\hbar \epsilon_{ijk} L_k$$

$$\boxed{[L_i, L_j] = -i\hbar \epsilon_{ijk} L_k}$$

The algebra of infinitesimal generators

The commutator of operators

$$[A, B] \equiv AB - BA$$

satisfies the Jacobi identity

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$$

We note that in general this is <u>not</u> <u>an associative algebra</u>. Particularly, if we impose associativity we get

$$[A,[A,B]] = [[A,A],B] = 0$$

For instance, considering the algebra of the previous generators

$$[X_i, X_j] = \epsilon_{ijk} X_k$$

we have explicitly

$$0 = [0, X_2] = [[X_1, X_1], X_2] \neq [X_1, [X_1, X_2]] = [X_1, X_3] = -X_2$$

Lie Algebra

Definition: A <u>Lie Algebra</u> is a vector space L over some field F endowed with a binary operation

$$[\,,\,] : L \times L \longrightarrow L$$

called the Lie bracket satisfying:

(i) bilinearity

$$[ax + by, z] = a[x,z] + b[y,z]$$
$$[z, ax + by] = a[z,x] + b[z,y]$$
$$\forall\ a, b \in F,\ x, y \in L$$

(ii) Jacobi identity

$$0 = [[x,y],z] + [[y,z],x] + [[z,x],y] \quad , \forall x,y,z \in L$$

(iii) antisymmetry

$$0 = [x,x] \quad , \quad \forall\ x \in L$$

Example: $L = \mathbb{R}^3$, $F = \mathbb{R}$

$$[x, y] \equiv x \times y \quad (\text{cross product in } \mathbb{R}^3)$$

For $x, y \in \mathbb{R}^3$, the Lie bracket is defined to be the usual cross product.

Checking:

i) $(ax + by) \times z = a(x \times z) + b(y \times z)$

ii) $(x \times y) \times z + (y \times z) \times x + (z \times x) \times y =$

$$= \{\epsilon_{ijk}\epsilon_{ilm} x_l y_m z_j + \epsilon_{ijk}\epsilon_{ilm} y_l z_m x_j + \epsilon_{ijk}\epsilon_{ilm} z_l x_m y_j\} \hat{e}_k$$

$$= (\epsilon_{ijk}\epsilon_{ilm} + \epsilon_{imk}\epsilon_{ijl} + \epsilon_{ilk}\epsilon_{imj}) x_l y_m z_j \hat{e}_k$$

$$= 0$$

iii) $x \times x = 0$

We may chose a basis for $\mathbb{R}^3$: $\mathcal{B} = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ – the canonical basis – and we have

$$[\hat{e}_i, \hat{e}_j] = \hat{e}_i \times \hat{e}_j = \epsilon_{ijk}\hat{e}_k$$