GROUP LINEAR REPRESENTATIONS

Thibes
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Definition. A matrix representation of a group G over a field K is a group homomorphism

$$T : G \longrightarrow GL_n(K)$$

We say n is the dimension of the representation.

Example 1: $G = \{\pm 1, \pm i, \pm j, \pm k\}$, $GL_2(\mathbb{C})$

$$1 \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad i \longrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad j \longrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Example 3 : $G = S_3 = \{ (), (12), (13), (23), (123), (321) \}$

$() \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad (12) \longrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$(23) \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\quad (13) \longrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\quad (123) \longrightarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

Example 4 (Regular Representation) : $G = V = \{ e, a, b, ab \}$

e	a	b	ab
a	e	ab	b
b	ab	e	a
ab	b	a	e

$a \rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

$b \rightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ $\quad ab \rightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

Every matrix A of order m with entries in the field K defines a linear transformation

$$x \longmapsto Ax$$

in the vector space K^m of column vectors over K. This correspondence is bijective.

Therefore we have the canonical group homomorphism

$$GL_m(K) \cong GL(K^m)$$

Now, generalizing the vector space K^m to an arbitrary vector space V we have the following more general definition.

Definition: Let G be a group. Given a vector space V over a field K, a linear representation of G in V is a group homomorphism

$$T : G \longrightarrow GL(V)$$

We say V is the representation space while the dimension of V is the dimension of the representation.

Recall $GL(V)$ is the group of all invertible linear transformations of V.

In case V has finite dimension n, each linear representation of G characterizes a class of matrix representations. In fact given a linear representation of G, by picking a basis

$$(e) = (e_1, \dots, e_n)$$

in V, every operator $T(g) \in GL(V)$ for $g \in G$ is described with respect to bases (e) by a matrix $T_{(e)}(g)$. The map

$$\begin{aligned} T_{(e)} : G &\longrightarrow GL_n(K) \\ g &\longmapsto T_{(e)}(g) \end{aligned}$$

is a matrix representation of G. By choosing another basis $(f) = (e)C$ with C a non singular matrix with entries in K, we get the relation

$$T_{(f)}(g) = C^{-1} \, T_{(e)}(g) \, C$$

In fact, $(e) = (e_1, \dots, e_m)$ and $(f) = (f_1, \dots, f_m)$ are two basis for the m dimensional vector space V over K. The non singular $m \times m$ matrix C changes from basis (e) to basis (f) in the sense

$$(f_1 \ f_2 \ \cdots \ f_m) = (e_1 \ e_2 \ \cdots \ e_m)\ C$$

or simply

$$(f) = (e)\ C$$

A vector $v \in V$ is represented in basis (e) by a column matrix

$$v_{(e)} = \begin{pmatrix} v_{(e)1} \\ v_{(e)2} \\ \vdots \\ v_{(e)m} \end{pmatrix}$$

The same vector $v \in V$ can be represented in basis (f) as

$$v_{(f)} = C^{-1} v_{(e)} = \begin{pmatrix} v_{(f)1} \\ \vdots \\ v_{(f)m} \end{pmatrix}$$

Note that we have

$$v = (e)\, v_{(e)} = (e)\, C\, C^{-1} v_{(e)} = (f)\, v_{(f)}$$

Thus for $T(g) \in GL(V)$ a linear operator, we take a generic vector $v \in GL(V)$ and write the corresponding matrix for $T(g)v \in V$ in basis (e) and (f) respectively as

$$(Tv)_{(e)} \quad \text{and} \quad (Tv)_{(f)} .$$

These two column matrices are related by

$$(Tv)_{(f)} = C^{-1} (Tv)_{(e)}$$

from which we get

$$\begin{aligned}
T_{(f)} v_{(f)} &= (Tv)_{(f)} \\
&= C^{-1} (Tv)_{(e)} \\
&= C^{-1} T_{(e)} v_{(e)} \\
&= C^{-1} T_{(e)} C v_{(f)} \qquad (\text{from } v_{(f)} = C^{-1} v_{(e)}) \\
&= \left(C^{-1} T_{(e)} C \right) v_{(f)}
\end{aligned}$$

and we have

$$\boxed{T_{(f)} = C^{-1} T_{(e)} C}$$

Definition: Two matrix representations T_1 and T_2 are said to be similar (denoted by $T_1 \simeq T_2$) when they have the same dimension and there exists a nonsingular matrix C such that

$$T_2(g) = C^{-1}\, T_1(g)\, C \quad , \quad \forall\, g \in G .$$

Definition: We say that two linear representations

$$T_1 : G \longrightarrow GL(V_1) \quad \text{and} \quad T_2 : G \longrightarrow GL(V_2)$$

are isomorphic or equivalent when there exists a vector space isomorphism $\sigma : V_1 \to V_2$, such that

$$\sigma\, T_1(g) = T_2(g)\, \sigma \quad , \quad \forall\, g \in G .$$

Note that, for a given $g \in G$, both $\sigma T_1(g)$ and $T_2(g)\sigma$ are applications from V_1 to V_2.

Example of Group Representation - Example 5

$$V = \mathbb{R}^2 \qquad G = (\mathbb{R}, +)$$

$$T : G \longrightarrow GL(V)$$

$$T : \mathbb{R} \longrightarrow GL(\mathbb{R}^2)$$

$$x \longmapsto T_x$$

$$T_x : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(a,b) \longmapsto (a+bx, b)$$

Note that

$$\begin{aligned} T_x(T_y(a,b)) &= T_x(a+by, b). \\ &= (a+by+bx, b) \\ &= (a+b(x+y), b) \\ &= T_{(x+y)}(a,b) \end{aligned}$$

Example 6:

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Let V denote the real linear space of all polynomials with real coefficients. To each $t \in \mathbb{R}$, assign the linear operator $L(t)$ by the rule

$$(L(t)f)(x) = f(x-t)$$

Note that $L(t)$ is linear because

$$\begin{aligned}(L(t)(f+g))(x) &= (f+g)(x-t)\\ &= f(x-t) + g(x-t)\\ &= (L(t)f)(x) + (L(t)g)(x)\end{aligned}$$

and $$\begin{aligned}(L(t)(\alpha f))(x) &= (\alpha f)(x-t)\\ &= \alpha f(x-t)\\ &= \alpha (L(t)f)(x)\end{aligned}$$

Note also that

$$\begin{aligned}(L(t+u)f)(x) &= f(x-t-u) = f((x-t)-u)\\ &= (L(u)f)(x-t)\\ &= L(t)(L(u)f)(x)\end{aligned}$$

That means

$$L(t+u) = L(t)\,L(u)$$

and L is a rep. of the additive group $(\mathbb{R},+)$.

<u>Definition</u>: Let $T : G \longrightarrow GL(V)$ be a linear representation of G in a vector space V. A subspace $U \subset V$ is invariant under T (or G-invariant) when

$$T(g)u \in U \quad , \forall\, g \in G \text{ and } u \in U$$

<u>Example 7</u>: Let L be the rep. of $(\mathbb{R},+)$ in the space of all polynomials defined by

$$\big(L(t)f\big)(x) = f(x-t)$$

For each natural $n \in \mathbb{N}$, the subspace of all polynomials of degree $\leq n$ is invariant

Definition A linear representation $T: G \to GL(V)$ is said to be irreducible if there are no nontrivial subspaces $U \subset V$ invariant under T.

Examples:

- Every one-dimensional rep. is irreducible.
- the rep. of $(\mathbb{R}, +)$ by rotations in the plane is irreducible.
- The rep of $(\mathbb{R}, +)$ by translations in the space of polynomials is not irreducible.

Definition A linear representation $T: G \to GL(V)$ is said to be completely reducible if every invariant subspace $U \subset V$ has an invariant complement W.

Recall: W is a complement of U if

$$V = U \oplus W$$